Game Theory, Information, Incentives

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Course # 320.501: Analytical Methods (part 5)

A General P-A-Model with Perfect (Symmetric) Information

- Model description
- Symmetric information contracts

$\rightarrow$ Macho-Stadler, ch. 2

Model description

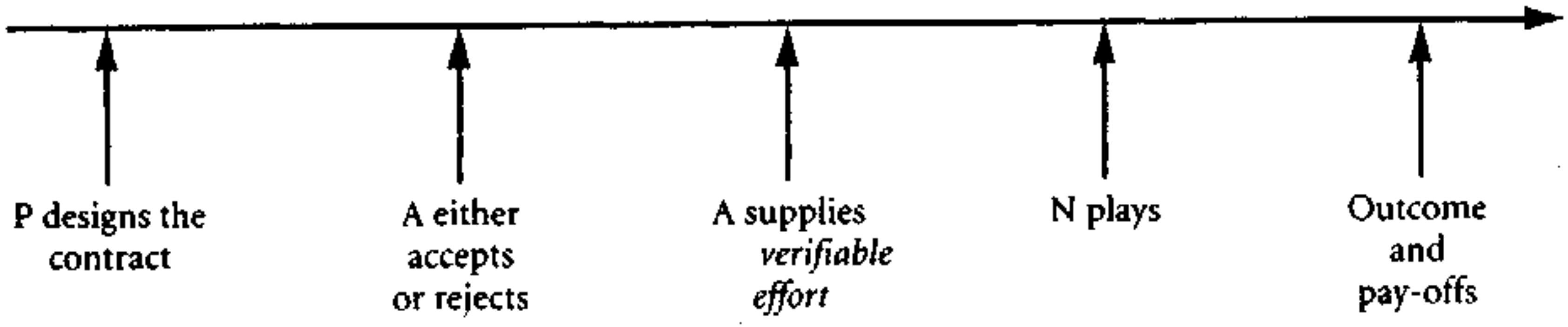
- Bilateral relationship (individuals, institutions, firms,...)
  - agent (A)
    - contracted by P to carry out some task
    - monetary value of result (production, outcome) $x \in X$
      - effort level (e) by A
      - random variable
  - principal (P)
    - designs and proposes contract to A
    - receives result x, pays agent w
    - P, A have same information (prior distribution) about x
  - contract (take it or leave it offer)
    - establishes relationship b/w P & A

      Document specifying obligations of P & A and transfers that must be made under all contingencies. Terms (obligations, transfers) and contingencies must be verifiable (judgeable by court).

- Result
  - $X = \{x_1, ..., x_n\}$, $\text{Prob}(x = x_i \,|\, e) = p_i(e) > 0$, $i \in \{1, 2, ..., n\}$
    $\sum_{i=1}^{n} p_i(e) = 1$
- Risk preferences
  - principal
    - $B(x - w)$, $B' > 0$, $B'' \leq 0$
  - agent
    - $U(w, e) = u(w) - v(e)$
    - $u'(w) > 0$, $u''(w) \leq 0$, $v(e) \geq 0$, $v'(e) > 0$, $v''(e) \geq 0$
  - conflict of interest
    - P interested in x, A is not
    - P not directly interested in e, A is
      greater effort makes better results more likely

Optimal symmetric information contracts (1P, 1A)

- Timing



- Contract: $(e, w(x_1), w(x_2), ..., w(x_n))$

Optimal symmetric information contracts (1P, 1A)

- P decides on: e, $\{w(x_i)\}_{i=1,\dots,n}$

$$\max_{e,\,\{w(x_i)\}_{i=1,\dots,n}} \sum_{i=1}^{n} p_i(e)\, B(x_i - w(x_i))$$

$$\text{s.t. } \sum_{i=1}^{n} p_i(e)\, u(w(x_i)) - v(e) \geq \underline{U}$$

  - e verifiable in base model
  - participation constraint

$$\mathcal{L} = \sum_{i=1}^{n} p_i(e)\, B(x_i - w(x_i)) + \lambda \left[\sum_{i=1}^{n} p_i(e)\, u(w(x_i)) - v(e) - \underline{U} \right]$$

  - well behaved for fixed e (why?)

Fix $e = e^o$ for the moment...

$\partial \mathcal{L} / \partial w(x_i) = 0$:

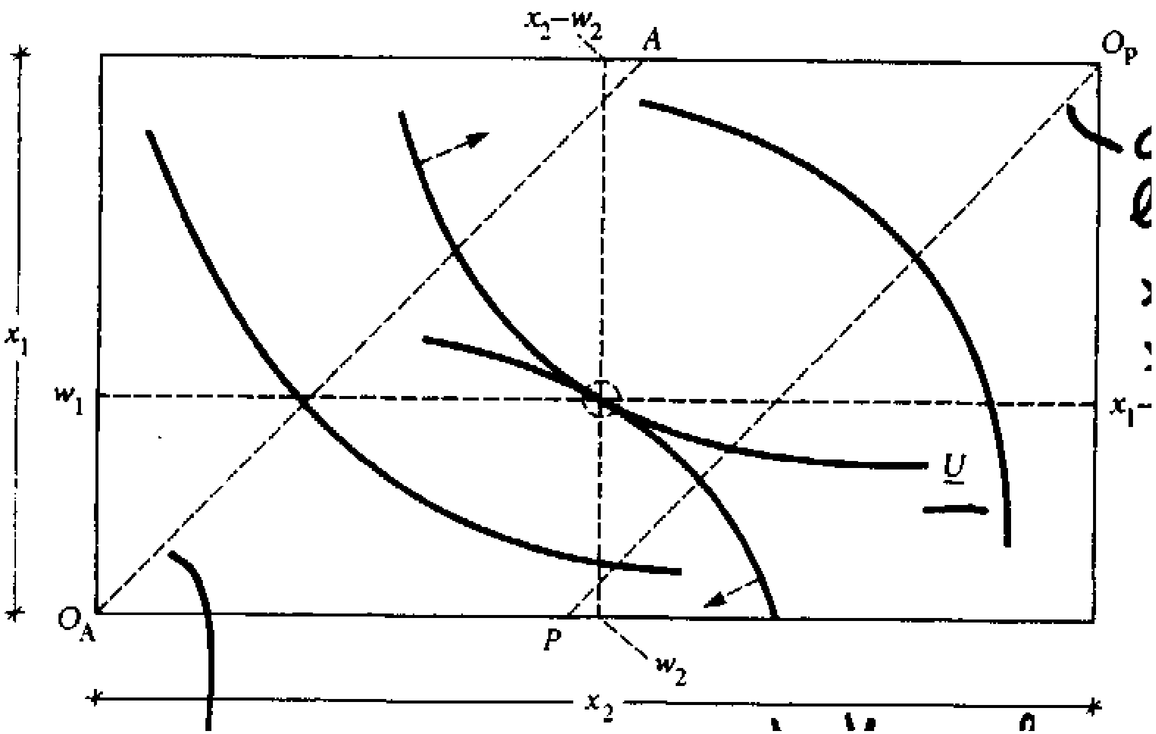
$$\lambda^o = \frac{B'(x_i - w^o(x_i))}{u'(w^o(x_i))}, \quad i = 1, 2, ..., n$$

- $\lambda^o > 0$ participation constraint binds

$$\frac{B'(x_i - w^o(x_i))}{u'(w^o(x_i))} = \lambda^o \Leftrightarrow \frac{B'(x_j - w^o(x_j))}{B'(x_i - w^o(x_i))} = \frac{u'(w^o(x_j))}{u'(w^o(x_i))}$$

- $MRS^P_{j,i} = MRS^A_{j,i}$ (Pareto efficiency) $\Rightarrow$ optimal risk sharing

Efficient risk sharing



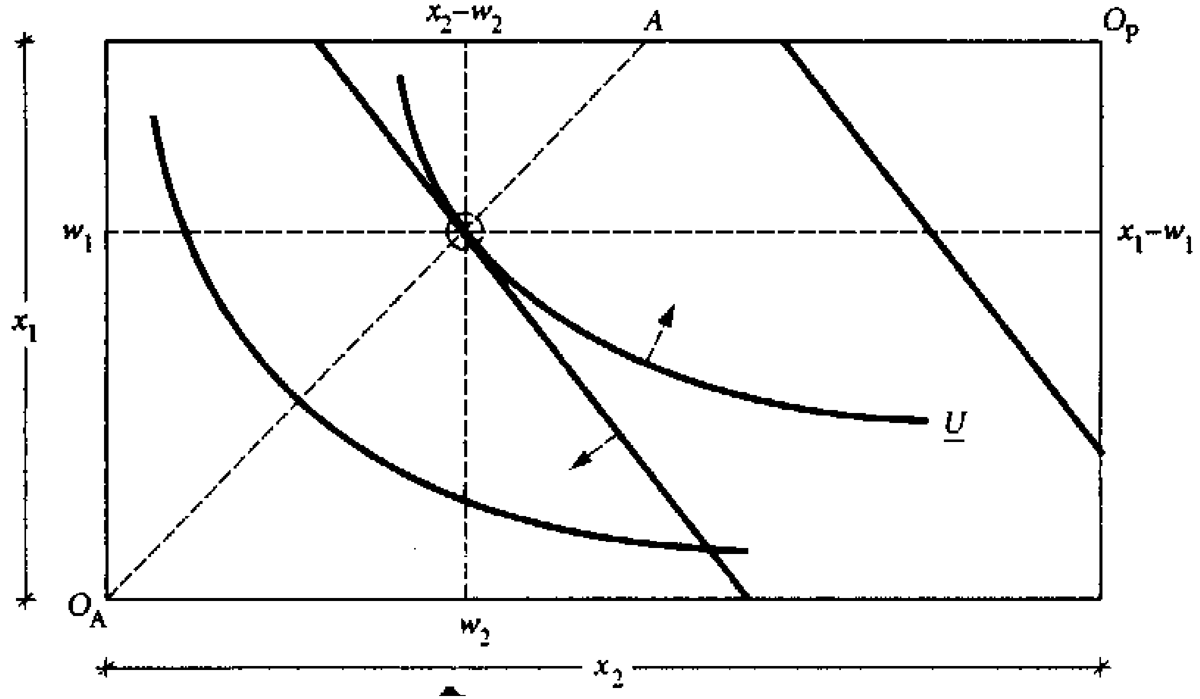
- risk = variability in result: $x_1 \neq x_2$, $x_1 - w_1 \neq x_2 - w_2$
- sure thing
  - certainty line P: $x_1 - w_1 = x_2 - w_2$
  - certainty line A: $w_1 = w_2$
- risk sharing: allocation b/w certainty lines (distribution of risk)

I. P risk-neutral, A risk-averse

- $u''(w_i) < 0$; $B'(x_i - w_i)$ constant $\Rightarrow B'(x_i - w_i)/B'(x_j - w_j) = 1$ $\Rightarrow u'(w_i)/u'(w_j) = 1 \Rightarrow w_i^o = w_j^o$!
- Optimal contract (distribution of risk)
  - P accepts all the risk, completely insuring A
  - A receives fixed wage w^o in all contingencies
  - $\sum_{i=1}^n p_i(e)\, u(w^o) - v(e) = \underline{U} \Rightarrow u(w^o) \sum_{i=1}^n p_i(e) - v(e) = \underline{U} \Rightarrow$ $u(w^o) - v(e) = \underline{U} \Rightarrow u(w^o) = \underline{U} + v(e)$

    $w^o = u^{-1}[\underline{U} + v(e)]$
  - w^o depends on e and $\underline{U}$ only (not on i)

I. P risk-neutral, A risk-averse



- Query: Determine the $MRS^P_{2,1}$ and $MRS^A_{2,1}$ in the equilibrium contract $(e^0, w(x_1), w(x_2))$.

II. P risk-averse, A risk-neutral

- $B''(x_i - w_i) < 0$; $u'(w_i)$ constant $\Rightarrow u'(w_i)/u'(w_j) = 1$

  $\Rightarrow B'(x_i - w_i)/B'(x_j - w_j) = 1 \Rightarrow x_i - w_i^o = x_j - w_j^o \equiv k$

- Optimal contract: franchise contract: $w^o(x_i) = x_i - k$
  - A accepts all the risk, completely insuring P
  - P receives fixed amount k in all contingencies
  - $\sum_{i=1}^{n} p_i(e)\,(x_i - k) - v(e) = \underline{U} \Rightarrow$
    $\sum_{i=1}^{n} p_i(e)\,x_i - k \sum_{i=1}^{n} p_i(e) = \underline{U} + v(e) \Rightarrow$
    $\sum_{i=1}^{n} p_i(e)\,x_i - k = \underline{U} + v(e)$

    $k = \sum_{i=1}^{n} p_i(e)\,x_i - \underline{U} - v(e)$
  - A “purchases” result from P

III. P risk-averse, A risk-averse

- P and A will need to accept part of risk
  - optimal distribution of risk $\leftarrow$ degrees of risk aversion
    - $r_p \equiv -B''/B'$, $r_a \equiv -u''/u'$

from FOC $-B'(x_i - w^o(x_i)) + \lambda\, u'(w^o(x_i)) = 0$:

$$-B'' \left[1 - \frac{d\,w^o}{d\,x_i}\right] + \lambda\, u'' \frac{d\,w^o}{d\,x_i} = 0$$

$$\frac{d\,w^o}{d\,x_i} = \frac{r_p}{r_p + r_a}$$

  - special cases: $r_p = 0$, $r_a = 0$

Optimal level of effort, e

- Suff. assumption (SOC): $\sum_{i=1}^{n} p_i''(e)\, x_i \leq 0$
- Backward induction: choose e, taking into account required change in $w(x_i)$
- Setting I: P risk-neutral, A risk-averse: $w^o = u^{-1}[\underline{U} + v(e)]$

$$\max_e \left[\sum_{i=1}^{n} p_i(e)\, x_i - u^{-1}[\underline{U} + v(e)] \right]$$

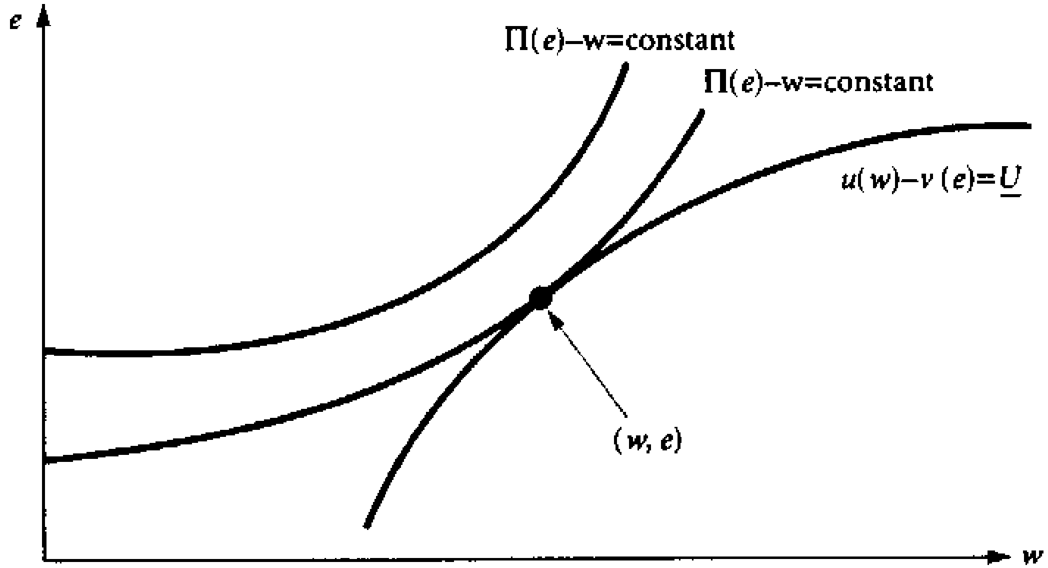
$$\sum_{i=1}^{n} p_i'(e^o)\, x_i = u^{-1\,\prime}[\underline{U} + v(e^o)]\, v'(e^o)$$

$$\sum_{i=1}^{n} p_i'(e^o)\, x_i = \frac{v'(e^o)}{u'(w^o)} \qquad \Rightarrow e^o$$

  - expected marginal profit of de = marginal increase in w P has to pay to A to compensate A for increased disutility of e ($MRS_{e,w}^{A}$)

P risk-neutral, A risk-averse

- $\Pi(e) \equiv \sum_{i=1}^{n} p_i(e)\, x_i - w \Rightarrow \Pi'(e) \equiv \sum_{i=1}^{n} p_i'(e)\, x_i$



  - identify direction of increasing utility (for P, for A)
  - in which sense do A and P have differing objectives?
  - calculate the slopes of the two curves at (w, e)

P risk-averse, A risk-neutral

- Setting II: P risk-averse, A risk-neutral ($u'' = 0$):
  $u(w^o(x_i)) = u(x_i - k) = x_i - k$
- Binding participation constraint:
  $\sum_{i=1}^{n} p_i(e)\,(x_i - k) - v(e) - \underline{U} = 0$
- P max $k = \sum_{i=1}^{n} p_i(e)\,x_i - v(e) - \underline{U} = 0$

$$\max_e \left[\sum_{i=1}^{n} p_i(e)\,x_i - v(e) - \underline{U} \right]$$

$$\sum_{i=1}^{n} p_i'(e^o)\,x_i = v'(e^o) \qquad \Rightarrow e^o$$

  - expected marginal profit of de = marginal cost of effort
    (notice that marginal profit received by A)