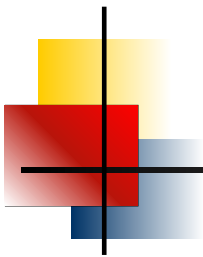


# *Game Theory, Information, Incentives*

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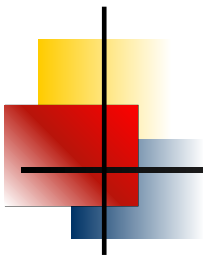
Course # 320.501: Analytical Methods (Part 1)



# *Economics, information, and incentives*

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- ▶ Business- & economic models:
  - strategic, multi-person decision problems
    - full information or symmetric information
    - asymmetric incomplete information
  
- ▶ Asymmetric information & differing objectives
  - hidden knowledge: adverse selection
  - hidden action: moral hazard
  - incentive problems b/c of opposing objectives of transaction partners
    - inefficiencies: mutually beneficial trades go unexploited
  
- ▶ Methodological frame: game theory



# Static Games of Complete Information

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## *Main learning objectives:*

- . basic coordinates & static games of complete information
  - ▶ Game theory: frameworks and basic concepts, language
  - ▶ Static games of complete information
    - strategic form games
    - dominant/dominated strategies
    - iterated elimination of strictly dominated strategies
    - **Nash equilibrium** (in pure strategies)
  - ▶ Applications (Cournot duopoly, Bertrand duopoly)
  - ▶ Mixed strategies
    - identification of mixed-strategy Nash equilibria
    - Application (batter-pitcher duel)



## *A beautiful mind...*

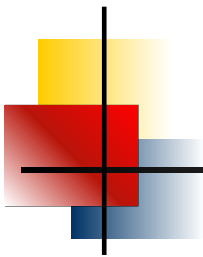
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John Nash (1928 – 2015)

Nobel Memorial Prize (1994), Abel Prize (2015)



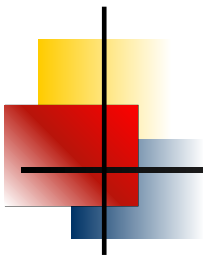
- ▶ Game theory: robust predictions in strategic, multi-person decision problems
- ▶ Nash equilibrium



## Example: Prisoner's dilemma

		Bob	
		fink	mum
Adam	fink	-10 , -10	0 , -20
	mum	-20, 0	-1 , -1

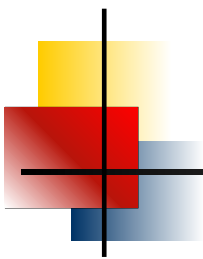
- ▶ ○ players {Adam, Bob}
- actions = {fink, mum}
- payoffs (matrix)
- ▶ strategic decision problem
- ▶ prediction of game theory: Nash equilibrium: (fink, fink) → worst outcome!  
(idea behind key-witness program)



## Setting the stage: basic coordinates

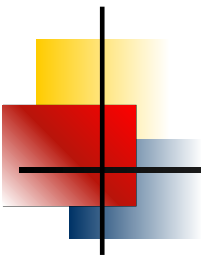
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- ▶ Game: every strategic multi-person decision problem
  - game theory (noncooperative)
  - basic ingredients: players, actions, payoffs
  
- ▶ Types of games
  - time
    - static (simultaneous-move) vs. dynamic (sequential-move)
  - information
    - complete information vs. incomplete information (Bayesian games)
    - perfect vs. imperfect information (dynamic games)
  
- ▶ incomplete information & information economics



► Solution concepts

	<b>complete information</b>	<b>incomplete information</b>
<b>static games</b>	iterated elimination of strictly dominated strategies; Nash equilibrium (NE)	Bayesian Nash equilibrium (BNE);
<b>dynamic games</b>	backward induction outcome; subgame perfect Nash equilibrium (SPNE)	Perfect Bayesian equilibrium; Sequential equilibrium



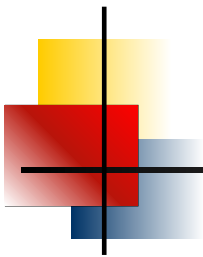
## ▶ Representation of games

- normal form = strategic form (matrix)
- extensive form (tree)

To every extensive form game there corresponds a strategic form game. For a given strategic form game there may be different corresponding extensive form games.

- frequently:
  - static games – normal form
  - dynamic games – extensive form





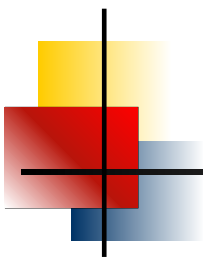
## ▶ Actions & strategies

- set of actions
  - static vs. dynamic games
  - finitely vs. infinitely many actions
- strategy: **complete contingent plan** that specifies a player's possible actions in every possible distinguishable circumstance.

		Player 2	
		L	R
Player 1	T	-10 , -10	0 , -20
	D	-20, 0	-1 , -1

## ▶ Player 1 plays first, then player 2 moves

- set of actions of 1:  $\{T, D\}$ ; of 2:  $\{L, R\}$
- strategies of 1:  $\{T, D\}$
- strategies of 2:  
 $\{(L \text{ if } 1 \text{ plays } T, L \text{ if } 1 \text{ plays } D), (L \text{ if } 1 \text{ plays } T, R \text{ if } 1 \text{ plays } D),$   
 $(R \text{ if } 1 \text{ plays } T, L \text{ if } 1 \text{ plays } D), (R \text{ if } 1 \text{ plays } T, R \text{ if } 1 \text{ plays } D) \}$



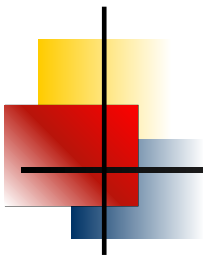
▶ strategies: pure vs. mixed

- sometimes it is in the best interest to mix pure strategies
  - pitcher-batter duel
- pure strategy vs. mixed strategy equilibria

▶ hierarchy of solution concepts

As games become progressively richer, equilibrium concepts need to be strengthened to rule out implausible equilibria.

- sequential equilibria  $\subseteq$  Bayesian Nash equilibria  $\subseteq$  subgame perfect NE  $\subseteq$  Nash equilibria
- stronger equilibrium concept always survives weaker concept



## Example: Prisoner's dilemma

		Bob	
		F	M
Adam	F	-10 , -10	0 , -20
	M	-20, 0	-1 , -1

2 players  $\{Adam, Bob\}$ , actions=strategies =  $\{F, M\}$

time: static (simultaneous move-, one shot-) game

information: complete and perfect

solution concept: (i) IESDS, (ii) NE

representation: normal form game

strategies: pure



# Static games of complete information

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## ▶ Strategic form games

- $N$  players  $i = 1, \dots, N$ , strategies  $s_i \in S_i$ , payoffs  $u_i$
- payoff function  $u_i : \times_{j=1}^N S_j \rightarrow \mathbf{R}$ ,  $S \equiv \times_{j=1}^N S_j \equiv S_1 \times S_2 \times \dots \times S_N$
- **definition:**  $G = \{S_1, S_2, \dots, S_n; u_1, u_2, \dots, u_n\}$ 
  - define  $G$  for the Prisoner's dilemma
  - define  $\times_{j=1}^N S_j$  for the Prisoner's dilemma
  - explain  $u_i$  for the Prisoner's dilemma

## ▶ Solution concept 1:

**Iterated elimination of strictly dominated strategies (IESDS)**



## A few notational conventions

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- ▶ individual strategy  $s_i$

set of individual strategies  $S_i, s_i \in S_i$

- ▶ specific joint strategy  $s = (s_1, s_2, \dots, s_N)$

joint strategy set  $S \equiv \times_{j=1}^N S_j, s \in S$

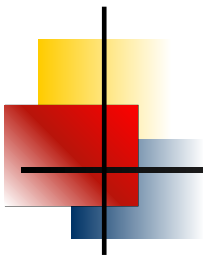
- ▶ “others” joint strategy  $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$

“others” joint strategy set

$S_{-i} = S_1 \times S_2 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_N, s_{-i} \in S_{-i}$

$(s_i, s_{-i}) = s!$

- demonstrate all of these concepts for the Prisoner’s dilemma



## Iterated elimination of strictly dominated strategies

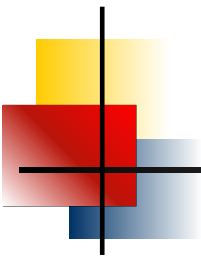
### ▶ Strictly dominant strategies

A strategy  $\hat{s}_i$ , for player  $i$  is **strictly dominant** if  $u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$  for all  $(s_i, s_{-i}) \in S$ .

- identify strictly dominant strategies in the following game

		Player 2	
		L	R
Player 1	U	3 , 0	0 , -4
	D	2 , 4	-1 , 8

- ▶ If  $\hat{s}_i$  exists,  $i$  plays it.



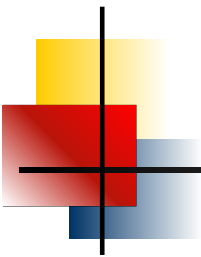
▶ Strictly **dominated** strategies

Player  $i$ 's strategy  $\hat{s}_i$  strictly dominates another of her strategies  $\bar{s}_i$ , if  $u_i(\hat{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$ . In this case,  $\bar{s}_i$  is strictly dominated in  $S$ .

If  $\bar{s}_i$  exists,  $i$  does not play it.

- identify strictly dominated strategies in the following game

		Player 2		
		l	m	r
Player 1	u	3 , 0	0 , -5	0 , -4
	c	1 , -1	3 , 3	-2 , 4
	d	2 , 4	4 , 1	-1 , 8

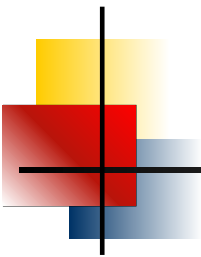


- ▶ Strictly dominated strategies can be iteratively eliminated from game
- ▶ Game theory – outcome: strategies that survive IESDS

		Player 2		
		l	m	r
Player 1	u	3 , 0	0 , -5	0 , -4
	c	1 , -1	3 , 3	-2 , 4
	d	2 , 4	4 , 1	-1 , 8

– calculate outcome according to IESDS



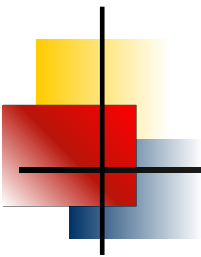


## ► Drawbacks

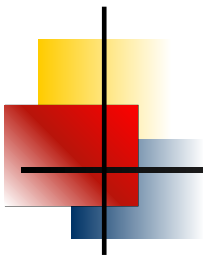
- rationality
- IESDS often not informative

		Player 2		
		l	m	r
Player 1	u	0 , 4	4 , 0	5 , 3
	c	4 , 0	0 , 4	5 , 3
	d	3 , 5	3 , 5	6 , 6

– all strategies survive IESDS



## ▶ Rationality experiment



## Nash equilibrium

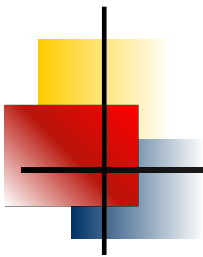
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players don't play SD strategies – what do they play?

▶ mutual best responses

▶ **Nash equilibrium:** Given  $G = (S_i, u_i)_{i=1}^N$ ,  $\hat{s} \in S$  is a pure strategy Nash equilibrium of  $G$  if for each player  $i$ ,  $u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{i-1})$  for all  $s_i \in S_i$ .

- NE is a joint strategy; observe the weak inequality
- find NE in the Prisoner's dilemma



find pure strategy Nash equilibria

		Player 2		
		l	m	r
Player 1	u	0 , 4	4 , 0	5 , 3
	c	4 , 0	0 , 4	5 , 3
	d	3 , 5	3 , 5	6 , 6

- ▶ **Theorem.** (i) If  $\hat{s}$  is a NE it survives IESDS.  
(ii) If only  $\hat{s}$  survives IESDS, then  $\hat{s}$  is the unique NE of the game.



## Example. Cournot duopoly (1838)

▶ Cournot game

▶  $P(Q) = a - Q$ ,  $Q = q_1 + q_2$ ,  $Q < a$ ,  $C_i(q_i) = cq_i$ ,  $0 < c < a$

$S_i = [0, \infty)$ , as for  $Q \geq a$ ,  $P(Q) = 0$ , no firm produces  $q_i \geq a$   
 $s_i = q_i \in S_i$ ,  $i = 1, 2$ ;  $s = (q_1, q_2) \in \times_{i=1}^2 S_i$ ;  $\times_{i=1}^2 S_i = \mathbf{R}_+^2$

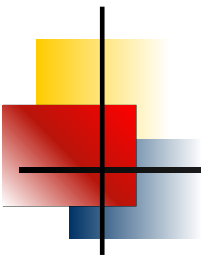
$$u_i(q_i, q_j) = q_i [P(q_i + q_j) - c] = q_i [a - q_i - q_j - c]$$

$$\text{NE: } u_i(\hat{s}_i, \hat{s}_j) \geq u_i(s_i, \hat{s}_j) \rightarrow u_i(\hat{q}_i, \hat{q}_j) \geq u_i(q_i, \hat{q}_j)$$

$$\max_{q_i \in S_i} q_i [a - q_i - \hat{q}_j - c]$$

$$\Rightarrow \hat{q}_1 = (a - \hat{q}_2 - c)/2 \text{ and } \hat{q}_2 = (a - \hat{q}_1 - c)/2$$

$$\Leftrightarrow \hat{q}_1 = \hat{q}_2 = (a - c)/3$$



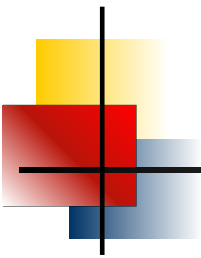
- ▶ prediction of game theory:  $\hat{s} = ((a - c)/3, (a - c)/3)$ 
  - unique pure strategy NE
  - $P(Q) = a - 2(a - c)/3 = a/3 + 2c/3 > c$  (as  $a > c$ )  
⇒ oligopoly profits
  - complete info vs. incomplete info (cost structure)  
→ Bayesian version of Cournot game



## Example. Bertrand duopoly (1883)

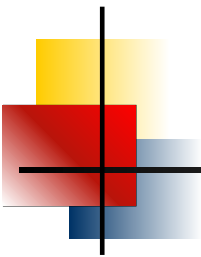
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- ▶ Bertrand game
  - ▶  $q_i(p_i, p_j) = a - p_i + b p_j$ ,  $2 > b > 0$ ; constant marginal cost  $c < a$
  - ▶  $p_i \in S_i = [0, \infty)$ , thus:  $s = (p_1, p_2) \in S = \mathbf{R}_+^2$
  - ▶  $u_i(p_i, p_j) = q_i(p_i, p_j)(p_i - c) = (a - p_i + b p_j)(p_i - c)$
  - ▶ NE:  $\max_{p_i \in S_i} (a - p_i + b \hat{p}_j)(p_i - c)$
- $\Rightarrow \hat{p}_1 = (a + b \hat{p}_2 + c)/2$  and  $\hat{p}_2 = (a + b \hat{p}_1 + c)/2$
- $\Leftrightarrow \hat{p}_1 = \hat{p}_2 = (a + c)/(2 - b)$



- ▶ prediction of game theory:  $\hat{s} = ((a + c)/(2 - b), (a + c)/(2 - b))$ 
  - unique pure strategy NE
  - Bertrand NE  $\neq$  Cournot NE:  $p_B > p_C$
  - $(a + c)/(2 - b) > c$  (as  $a > c$ )  
 $\Rightarrow$  oligopoly profits
  - complete info vs. incomplete info (cost structure)  
 $\rightarrow$  Bayesian version of Bertrand game





▶ Film Clip (A beautiful mind)

▶ Discussion:

Is the strategy “No one for the blonde” Pareto efficient?

Is it a Nash equilibrium?



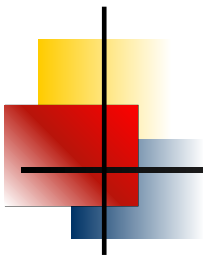
## Mixed strategies

- ▶ **Mixed strategies:** Consider a finite  $G = (S_i, u_i)_{i=1}^N$ . A mixed strategy for  $i$  is a probability distribution,  $m_i$ , over  $S_i$ . That is,  $m_i : S_i \rightarrow [0, 1]$ ,  $0 \leq m_i(s_i) \leq 1$  and  $\sum_{s_i \in S_i} m_i(s_i) = 1$ .
- ▶ Set of  $i$ 's mixed strategies:  
$$M_i \equiv \{m_i : S_i \rightarrow [0, 1] \mid \sum_{s_i \in S_i} m_i(s_i) = 1\}$$
  - simplex ( $\rightarrow$  show simplex for 2 or 3 strategies)

pure strategies  $\subset$  mixed strategies
- ▶ Payoffs: **expected**  $u$ -function
  - $s = (s_1, s_2, \dots, s_N)$
  - probability of  $s \in S$ :  $m_1(s_1) m_2(s_2) \dots m_N(s_N)$

if strategies are chosen independently,  $\text{prob}(s) =$  product of probabilities  $m_i(s_i)$

complication: not only  $i$  randomizes, but so do all others as well



$$\blacktriangleright u_i(m) = \sum_{s \in S} \underbrace{m_1(s_1) m_2(s_2) \dots m_N(s_N)}_{\text{probability of } s} u_i(s)$$

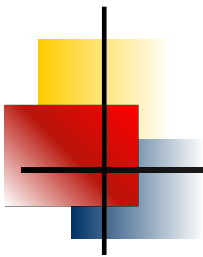
$$m = (m_1, m_2, \dots, m_N)$$

$$m \in M \equiv M_1 \times M_2 \times \dots \times M_N$$

$$m_{-i} = (m_1, m_2, \dots, m_{i-1}, m_{i+1}, \dots, m_N)$$

$$M_{-i} \equiv M_1 \times M_2 \times \dots \times M_{i-1} \times M_{i+1} \times \dots \times M_N$$

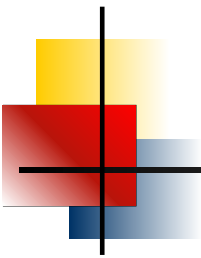
$$m_{-i} \in M_{-i}$$



## Example

		Player 2	
		l (pr=p)	r (pr=1-p)
Player 1	u (pr=q)	1 , 2	3 , 0
	d (pr=1-q)	2 , 4	1 , 1

- ▶  $m_1(u) = q, m_1(d) = (1 - q), m_2(l) = p, m_2(r) = (1 - p),$   
 $m = (m_1, m_2)$
- ▶ expected payoff of playing  $u$ :  $p \cdot 1 + (1 - p) \cdot 3$ ;  $d$ :  $p \cdot 2 + (1 - p) \cdot 1$
- ▶ player 1's expected payoff of  $m_1$ , given  $m_2$ :  
 $u_1(m) = q [p \cdot 1 + (1 - p) \cdot 3] + (1 - q) [p \cdot 2 + (1 - p) \cdot 1]$
- ▶ calculate  $u_2(m)$



▶ Interpretation of mixed strategies

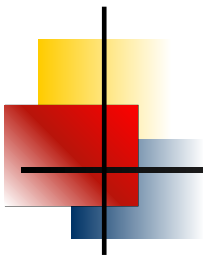
- sometimes in one's best interest to employ randomization mechanism

→ expected payoff of  $m_i > s_i$ ,  
for probabilities chosen so to maximize expected payoff

▶ Mixed strategy Nash equilibrium

- Given  $G = (S_i, u_i)_{i=1}^N$ ,  $\hat{m} \in M$  is a Nash equilibrium of  $G$  if for each player  $i$ ,  $u_i(\hat{m}) = u_i(\hat{m}_i, \hat{m}_{-i}) \geq u_i(m_i, \hat{m}_{-i})$  for all  $m_i \in M_i$ .

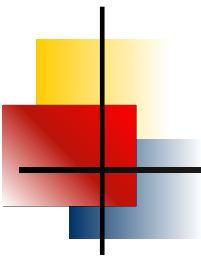
– infinitely many strategies to be tested!



► Batter-pitcher duel (baseball)

		batter	
		F	C
pitcher	F	-1 , 1	1 , -1
	C	1, -1	-1 , 1

- no SD strategies, no NE
- how to find mixed strategy NE,  $\hat{m}$ :  
$$u_i(\hat{m}) = u_i(\hat{m}_i, \hat{m}_{-i}) \geq u_i(m_i, \hat{m}_{-i}) \quad \forall m_i \in M_i, \forall i$$
- mixed strategy NE test

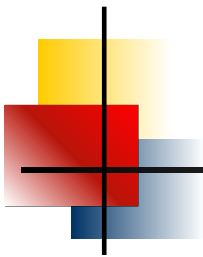


► **Theorem.** Mixed strategy Nash equilibrium test.

For all  $i$ , the following holds:

- $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$  for all  $s_i$  played with positive probability.
- $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$  for all  $s_i$  played with zero probability.
- batter-pitcher duel:  $S_i = \{F, C\}$ , both strategies played with positive probability:  
 $u_i(F, \hat{m}_{-i}) = u_i(C, \hat{m}_{-i})$

→ calculate mixed strategy NE for batter pitcher duel



## Three important results

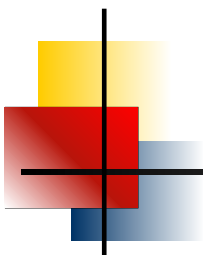
- ▶ **Proposition.** A pure strategy of  $i$  can be a best response to a mixed strategy by  $j$ , even if it is not a best response to any other pure strategy by  $j$ .

		Player 2	
		L	R
Player 1	T	3 , -	0 , -
	M	0 , -	3 , -
	B	2 , -	2 , -

- Let  $m_2 = (1/2, 1/2)$ , and  $s_1 = B$ .

Then,  $s_1$  is a best response to  $m_2$ :  
 $E[B] = 2 > E[M] = E[T] = 3/2$ .



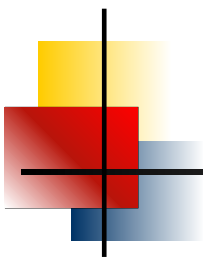


- ▶ **Proposition.** A pure strategy can be dominated by a mixed strategy, even if it is not dominated by any other pure strategy.

		Player 2	
		L	R
Player 1	T	6 , -	0 , -
	M	0 , -	6 , -
	B	2 , -	2 , -

- $B$  is not dominated by either  $T$  or  $M$ . Let  $m_1 = (1/2, 1/2, 0)$ , and  $s_1 = B$ .

Then, for all  $m_2$ :  $u_1(s_1, m_2) = 2 < 3 = u_1(m_1, m_2)$ .



► **Theorem.** In any finite  $G$ , there exists a Nash equilibrium.

○ *Proof sketch.* (1) Define best-response correspondence and show that any fixed-point of correspondence is a NE;

(2) By Kakutani's fixed-point theorem, best-response correspondence has a fixed point.

(1) For any given  $m$ , define player  $i$ 's set of best responses  $\phi_i(m) \subseteq M_i$

$\phi(m) \equiv \times_{i=1}^N \phi_i(m)$  (best response correspondence)

$m^*$  is a fixed point of  $\phi(m)$  if  $m^* \in \phi(m)$

$m^* = \hat{m}$ , obviously.

(2) As  $m \in M$ , the domain of  $\phi(m)$  is nonempty, compact, convex.

$\phi(m)$  is upper hemicontinuous from  $M$  into  $M$ .

By Kakutani's fixed point theorem, there exists  $m^* \in M$  such that  $m^* \in \phi(m)$ . ||