Game Theory, Information, Incentives

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Course # 320.501: Analytical Methods (part 3)

Simultaneous Move Bayesian Games

- Bayesian Game Environment
- Examples (static games under incomplete information)
  - Battle of sexes
  - Entry game - version 1
  - Entry game - version 2
- Formalizing Bayesian Games
- Sheriff's Dilemma
- Fight Game
- A Public Good Game

Bayesian Game Environment

- Incomplete information about others' payoffs or types
  - strategic situation: i's payoff depends on other players' actions
  - other players' actions depend on their respective types
  - i know other players but not the specific types involved in game
    $\rightarrow$ i does not know which game is played
- Wide variety of incomplete information scenarios
  - players are of hidden types
    (cournot duopoly with incomplete information)
    - private information ∴ public information (common knowledge)
  - both players do not know the state of nature
    (selling a house - seller wants to sell when economy is"good",
    buyer wants to buy when economy is "bad",
    state of economy is not known to any of players)

Example: Battle of Sexes (w/ incomplete information)

- **Players**: 1 (the lady); 2 (the man):
  the lady wants to go out with the man

  2 has two types: X (wants to go out with her);
  Y (wants to avoid her)
  1 has incomplete information about which type 2 is

- **Actions**: Go to music performance, either Bach- or Stravinski concert

  Player 1 prefers Bach (B); 2 prefers Stravinski (S)

- **Information**: common knowledge: both players know the preferences (B,S) of the other one; both have *same prior beliefs*

  private information: each player knows only her/his type
  (after nature played)

- Query. In which way does this game differ from a prisoner's dilemma game?

Battle of Sexes: Strategies and Payoffs

Player 2 (type X)

Player 1		B	S
	B	2 , 1	0 , 0
	S	0 , 0	1 , 2

Player 2 (type Y)

Player 1		B	S
	B	2 , 0	0 , 2
	S	0 , 1	1 , 0

- types $\theta_i \in \Theta_i$
- (pure) strategy $s_i(\theta_i) : \Theta_i \to S_i$

  $S_i = \{B, S\}$, $\Theta_2 = \{X, Y\}$

  $s_1 =?$, $s_2 =?$
- (prior) belief $p(X) = 1/2$, $p(Y) = 1/2$ $\quad \to$ not clear which game is played

Pure Strategy Bayesian Nash Equilibrium

Player 2

Player 1		(B,B)	(B,S)	(S,B)	(S,S)
	B	2 , (1,0)	1 , (1,2)	1 , (0,0)	0 , (0,2)
	S	0 , (0,1)	1/2 , (0,0)	1/2 , (2,1)	1 , (2,0)

- BNE (pure strategies) $= \{B, (B, S)\}$
- different beliefs - probably different BNE

Entry Game - version 1

High cost incumbent (player 1)

		2: Enter	2: Not Enter
1	Build	0 , -1	2 , 0
1	Not Build	2 , 1	3 , 0

Low cost incumbent (player 1)

		2: Enter	2: Not Enter
1	Build	3 , -1	5 , 0
1	Not Build	2 , 1	3 , 0

- Information:

  Prior: high (low) cost with probability p_1 (with $(1 - p_1)$)

  1 knows that 2 knows about his strictly dominating strategies

- 2: enter if expected payoff from entering $(= 2p_1 - 1) >$ from not entering $(= 0)$

BNE

- If $p_1 > 1/2$

  BNE $=$ {(Not Build if high, Build if low), Enter}

- If $p_1 < 1/2$

  BNE $=$ {(Not Build if high, Build if low), Not Enter}

- If $p_1 = 1/2$ any of the above

Entry Game - version 2

High cost incumbent (player 1)

		2: Enter	2: Not Enter
1	Build	0 , -1	2 , 0
	Not Build	2 , 1	3 , 0

Low cost incumbent (player 1)

		2: Enter	2: Not Enter
1	Build	3/2 , -1	7/2 , 0
	Not Build	2 , 1	3 , 0

- Information

  Prior: high (low) cost with probability p_1 (with $(1 - p_1)$)

  1 knows that 2 knows about his one strictly dominating strategy

- Probabilities to be set by players

  $p(\text{Enter}) = y$, $p(\text{Not Enter}) = 1 - y$

  $p(\text{Build}) = x$, $p(\text{Not Build}) = 1 - x$ for low cost type

- Equilibrium beliefs $= x, y$

Best Responses

- Player 2

expected payoff from Enter:
$1\,p_1 + (1-p_1)(-x + 1\,(1-x)) = 1 - 2x(1-p_1)$

expected payoff from Not Enter $= 0$

$$\begin{aligned} y = 1 \text{ (Enter)} \quad & \text{if} \quad x < 1/[2(1-p_1)] \\ y \in [0,1] \quad & \text{if} \quad x = 1/[2(1-p_1)] \\ y = 0 \text{ (Not Enter)} \quad & \text{if} \quad x > 1/[2(1-p_1)] \end{aligned}$$

Best Responses

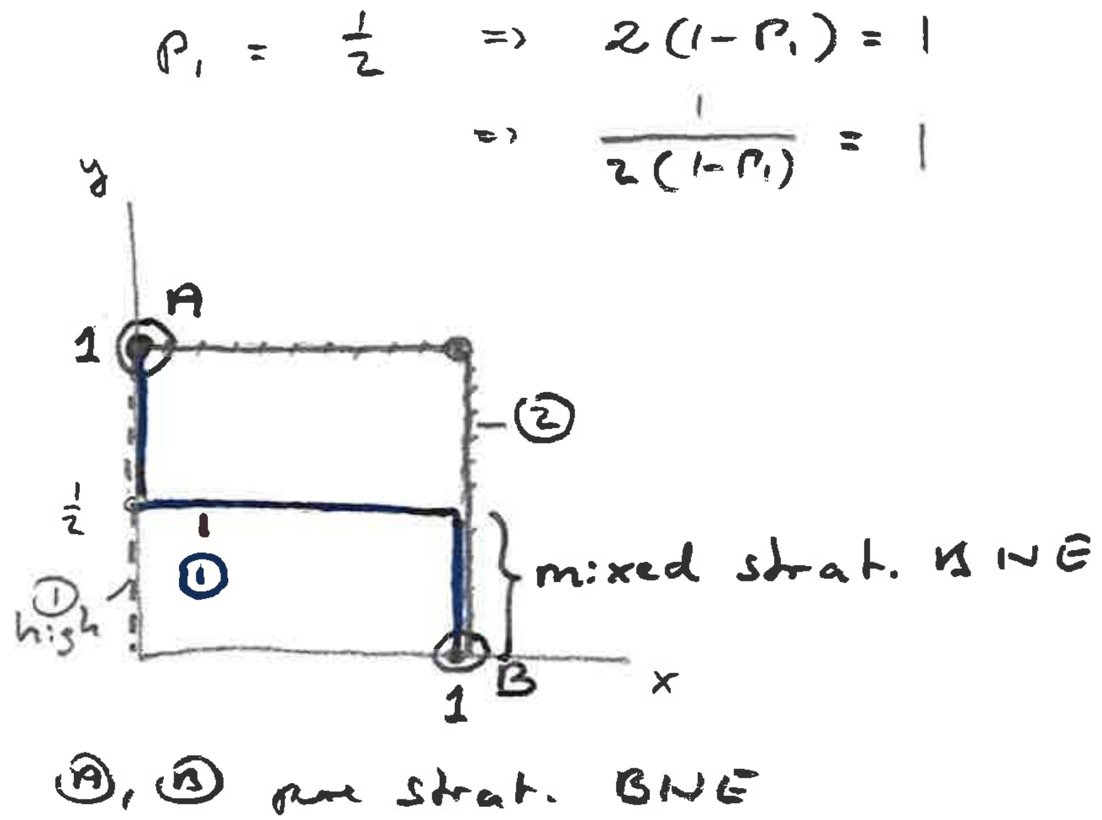
- Player 1 (high type): $x = 0$ (Not Build) for all $y \in [0, 1]$
- Player 1 (low type)

expected payoff from Build: $3/2y + 7/2(1 - y) = 7/2 - 4/2y$
expected payoff from Not Build $= 2y + 3(1 - y) = 3 - y$

$$
\begin{aligned}
x = 1 \text{ (Build)} \quad & \text{if} \quad y < 1/2 \\
x \in [0, 1] \quad & \text{if} \quad y = 1/2 \\
x = 0 \text{ (Not Build)} \quad & \text{if} \quad y > 1/2
\end{aligned}
$$

BNE

- (x, y) as mutual best responses, given p_1
- Let $p_1 = 1/2$



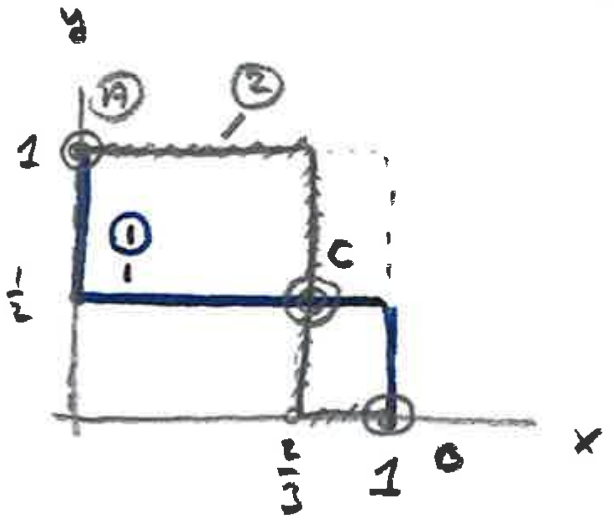
- Identify the pure strategy BNE

BNE

- Let $p_1 = 1/4$

$$p_1 = \frac{1}{4} \Rightarrow 2(1-p_1) = \frac{3}{2}$$

$$\Rightarrow \frac{1}{2(1-p_1)} = \frac{2}{3}$$



ⓒ mixed strat. BNE

- Identify the pure- and mixed strategy BNE
- Suppose belief is that $p_1 > 1/2$, what happens to the BNE?

Formalizing Bayesian Games

- Players $i = 1, ..., N$
- Types
  - type of player i is $\theta_i \in \Theta_i$ (i observes own type only)
  - specific type profile $\theta = (\theta_i,\, \theta_{-i}) \in \Theta \equiv \Theta_1 \times \Theta_2 \times \cdots \times \Theta_N$
- Probabilities (beliefs)
  - prior (joint distribution) $p(\theta)$, $p(\theta_i, \theta_{-i})$
  - marginal distribution $p(\theta_i)$
  - conditional distribution $p(\theta_{-i}|\theta_i) = p(\theta_i, \theta_{-i})/p(\theta_i)$

    (Bayes' rule)

What are the prior- and conditional distributions in the entry game?

Formalizing Bayesian Games

- Pure actions $s_i \in S_i$

  (Pure) strategies $s_i(\theta_i) : \Theta_i \to S_i$

  Show pure actions and strategies for the entry game

- Payoffs
  - $u_i(s_i, s_{-i}, \theta_i, \theta_{-i})$
  - same strategies may yield different payoffs for different type profile

    ($\to$ entry game)

    players don't know which game is being played

Formalizing Bayesian Games

- Bayesian Nash equilibrium (in pure strategies)

  For each i and each θ_i:

$$s_i^*(\theta_i) \in \arg\max_{s_i \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i)\, u_i(s_i, s_{-i}^*, \theta_i, \theta_{-i}) \qquad (1)$$

  - BNE is a pure action for each type of each player
  - Demonstrate (1) for the BNE $((NB, NB), E), p_1 = 1/2$.

Sheriff's dilemma

- Players and types: player 1: suspect; player 2: Sheriff

  $\theta_1 \in \Theta_1 = \{c \text{ (criminal) }, nc \text{ (not criminal)}\}$,
  $\theta_2 \in \Theta_2 = \{S \text{ (Sheriff)}\}$

  $\Theta = \Theta_1 \times \Theta_2 = \{(c, S), (nc, S)\}$

- Probabilities (beliefs)
  - prior probability distribution: $p((c, S)) = q$, $p((nc, S)) = 1 - q$
  - marginal distribution: $p(c) = q$, $p(nc) = (1 - q)$, $p(S) = 1$
  - conditional probabilities:

    $p(c|S) = p(c, S)/p(S) = q/1 = q$

    $p(nc|S) = p(nc, S)/p(S) = (1 - q)/1 = 1 - q$

    $p(S|c) = p(c, S)/p(c) = q/q = 1$

    $p(S|nc) = p(nc, S)/p(nc) = (1 - q)/(1 - q) = 1$

Sheriff's Dilemma

- Pure actions $s_i \in S_i = \{s \text{ (shoot) }, ns \text{ (not shoot)}\}$, $i = 1, 2$
- Payoffs

Player 1 - criminal (c type)

Player 2 (columns), Player 1 (rows)

	s	ns
s	0 , 0	2 , -2
ns	-2 , -1	-1 , 1

Player 1 - not criminal (nc type)

Player 2 (columns), Player 1 (rows)

	s	ns
s	-3 , -1	-1 , -2
ns	-2 , -1	0 , 0

Sheriff's Dilemma: BNE

- Player 1

  $\theta_1 = c$: s strictly dominates ns

  $\theta_1 = nc$: ns strictly dominates s

  $\to s_1^*(\theta_1) = (s, ns)$

- expected payoffs for player 2 according to (1):

  s:

  $$q \underbrace{u_2(s, s^*, S, c)}_{u_i(s_i, s^*_{-i}, \theta_i, \theta_{-i})} + (1-q)\, u_2(s, ns^*, S, nc) = 0\, q + (1-q)(-1) = q - 1$$

  ns:

  $$q\, u_2(ns, s, S, c) + (1-q)\, u_2(ns, ns, S, nc) = -2q + (1-q)0 = -2q$$

  $$\to s_2^*(\theta_2) = \begin{cases} s\,, & q > 1/3 \\ ns\,, & q < 1/3 \\ \text{any mix}\,, & q = 1/3 \end{cases}$$

- Bayesian Nash equilibrium: $s^* = (s_1^*(\theta_1), s_2^*(\theta_2))$

Fight Game

- Player 1 (strong, weak), Player 2, $S_i = \{f(\text{fight}), nf(\text{not fight})\}$

Player 1 - strong (s type)

		2: f	2: nf
1	f	1 , -2	2 , -1
	nf	-1 , 2	0 , 0

Player 1 - weak (w type)

		2: f	2: nf
1	f	-2 , 1	2 , -1
	nf	-1 , 2	0 , 0

- $p(s) = q$, $p(w) = 1 - q$
- not a strictly dominating strategy for all types of player 1
- $s_1^*(\theta_1) = (f, ?)$

Expected Payoffs

- Probabilities (beliefs)
  - player 1 (w): $p(f) = x$, $p(nf) = 1 - x$
  - player 2: $p(f) = y$, $p(nf) = 1 - y$
  - pure strategy: $x, y \in \{0, 1\}$
- Expected payoffs
  - Player 1 decides on x

    $\theta_1 = s$: f for all $y \in [0, 1]$

    $\theta_1 = w$:

    f: $y(-2) + (1 - y)2$

    nf: $y(-1) + 0$

    $x = 1$ if expected payoff from f exceeds that from nf: $y < 2/3$

Best responses, BNE

- Expected payoffs
  - Player 2 decides on y

    f: $q(-2) + (1-q)[x1 + (1-x)2]$

    nf: $q(-1) + (1-q)[x(-1) + 0]$

    $y = 1$ if expected payoff from f exceeds that from nf: $q < 2/3$

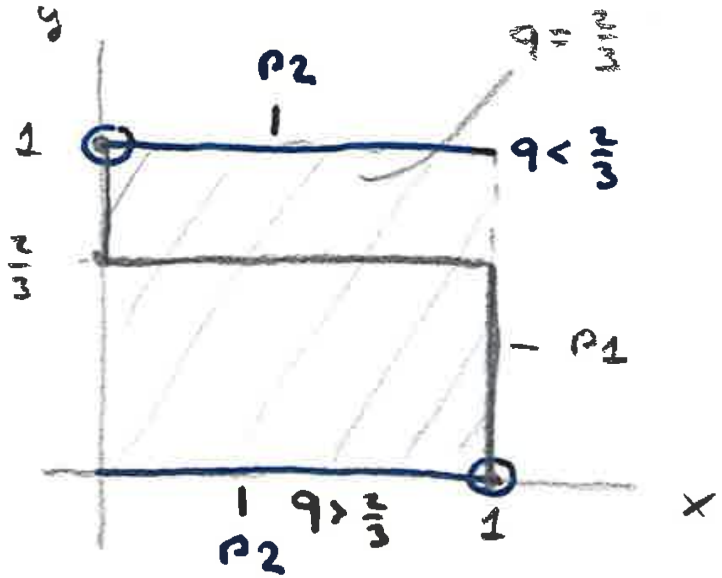
- Best responses

$$s_1^*(w) = \begin{cases} x = 1\,, & y < 2/3 \\ x = 0\,, & y > 2/3 \\ x \in [0,1]\,, & y = 2/3 \end{cases} \qquad s_2^* = \begin{cases} y = 1\,, & q < 2/3 \\ y = 0\,, & q > 2/3 \\ y \in [0,1]\,, & q = 2/3 \end{cases}$$

- Bayesian Nash equilibrium: $s^* = s_1^*(\theta_1), s_2^*(\theta_2))$

BNE

optimal choice of y does not depend on x, only on belief q



- BNE in pure strategies
  - $s_1^*(\theta_1) = (f, nf)$, $s_2^*(\theta_2) = f$, $q < 2/3$
  - $s_1^*(\theta_1) = (f, f)$, $s_2^*(\theta_2) = nf$, $q > 2/3$

Bayesian Games with a Continuum of Types

- : A Public Good Game

  - 2 players; contribute/do not contribute to a public good
  - private cost of contributing: $c_i \in [\underline{c}, \overline{c}] = \Theta_i$, where $\underline{c} < 1 < \overline{c}$

    c_i is type of player i - there is a continuum of types
  - benefit if public good is provided $= 1$ for both i
  - pure actions $s_i \in S_i = \{0, 1\}$, $i = 1, 2$ where 1 = contribute, 0 = not contribute

- Payoffs

		Player 2: 1	Player 2: 0
Player 1	1	$(1 - c_1)$, $(1 - c_2)$	$(1 - c_1)$, 1
	0	1 , $(1 - c_2)$	0 , 0

Public Good Game

- Pure strategy: $s_i(c_i) : [\underline{c}, \overline{c}] \to \{0, 1\}$
- BNE: pair of such maps $(s_1^*(c_1), s_2^*(c_2))$:

$$s_i^*(c_i) \in \arg\max_{s_i \in \{0,1\}} E|_{c_j} u_i(s_i, s_j^*(c_j), c_i)$$

where $u_i(s_i, s_j, c_i) = \max\{s_i, s_j\} - c_i\, s_i$

- Probabilities (beliefs) - to be calculated: $z_j = Pr(s_j^*(c_j) = 1)$
- Player 1
  - $E|_{c_2} u_1(1, s_2^*(c_2), c_1) = z_2(1 - c_1) + (1 - z_2)(1 - c_1) = 1 - c_1$
  - $E|_{c_2} u_1(0, s_2^*(c_2), c_1) = z_2\, 1 + (1 - z_2)\, 0 = z_2$

Public Good Game

- $s_1^*(c_1) = 1$ if $(1 - c_1) > z_2 \Leftrightarrow c_1 < 1 - z_2 \equiv c_1^*$ (cutoff level)

  $s_1^*(c_1) = 0$ if $c_1 > c_1^*$

- let F denote the CDF: $z_j = Pr(s_j^*(c_j) = 1) = Pr(c_j \leq c_j^*) = F(c_j^*)$

  $\Rightarrow c_1^* = 1 - F(c_2^*)$ and $c_2^* = 1 - F(c_1^*)$

  $c_1^* = c_2^* = c^*$ (by symmetry)

  $\Rightarrow$ solution: $c^* = 1 - F(c^*)$

Public Good Game

- Bayesian Nash equilibrium

$$s_1^*(c_1) = \begin{cases} 1\,, & c_1 < c^* \\ 0\,, & c_1 > c^* \\ any\,, & c_1 = c^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} 1\,, & c_2 < c^* \\ 0\,, & c_2 > c^* \\ any\,, & c_2 = c^* \end{cases}$$

- Example: uniform distribution: $F(c^*) = (c^* - \underline{c})/(\overline{c} - \underline{c})$

$$\Rightarrow c^* = 1 - (c^* - \underline{c})/(\overline{c} - \underline{c}) = \overline{c}/(1 + \overline{c} - \underline{c})$$

For $\overline{c} = 2, \underline{c} = 0 \Rightarrow c^* = 2/3$:

$$s_1^*(c_1) = \begin{cases} 1\,, & c_1 < 2/3 \\ 0\,, & c_1 > 2/3 \\ any\,, & c_1 = 2/3 \end{cases} \qquad s_2^*(c_2) = \begin{cases} 1\,, & c_2 < 2/3 \\ 0\,, & c_2 > 2/3 \\ any\,, & c_2 = 2/3 \end{cases}$$