Game Theory, Information, Incentives

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Course # 320.501: Analytical Methods (part 4)

Sequential equilibrium

- Behavioral strategies (b_i)
- Subgame perfection $\rightarrow$ reasonable strategies?
- Sensible system of beliefs for given behavioral strategies
- Consistent assessment
- Sequential rationality: b_i as best responses
- Sequential equilibrium

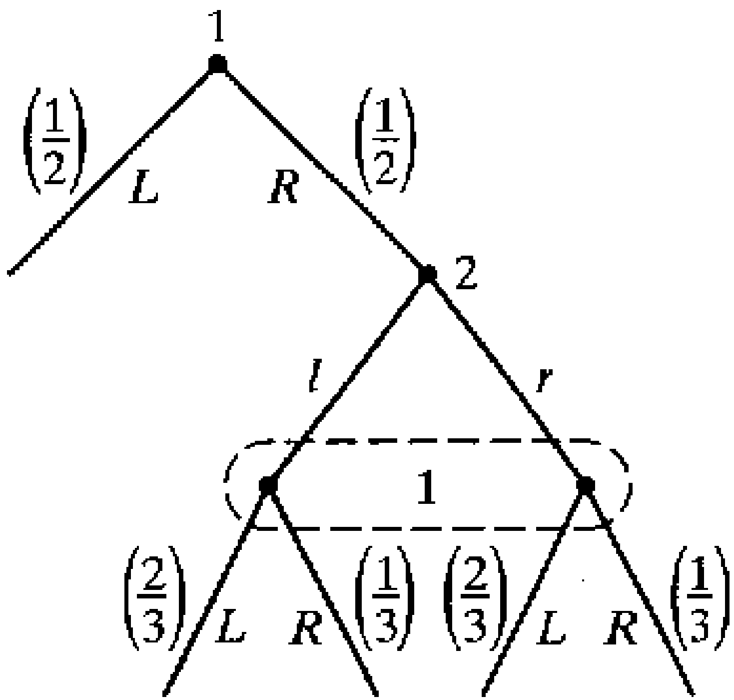
$\rightarrow$ Jehle/Reny, Chp.7

Behavioral strategies

- Mixed strategies vs. behavioral strategies

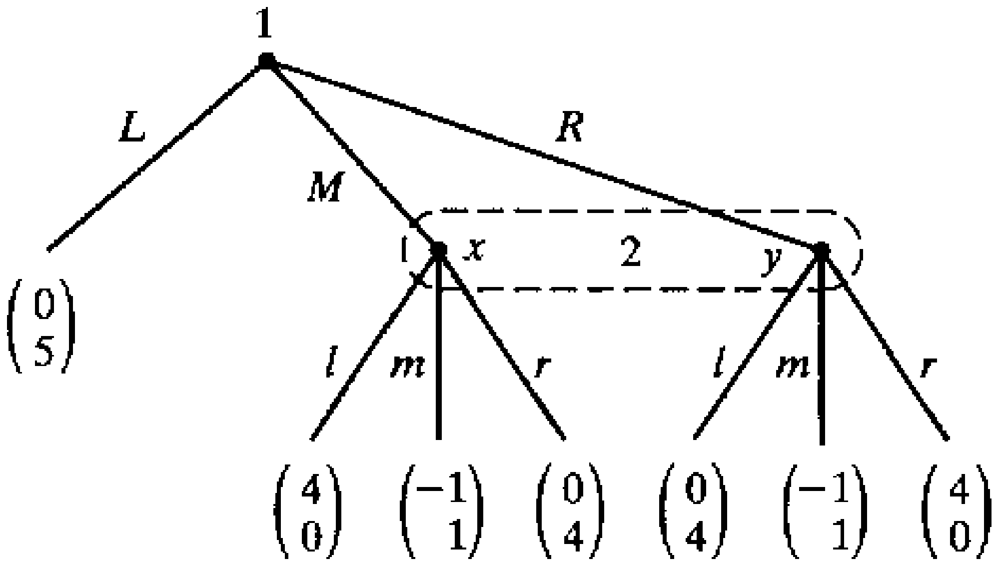
  $b_i(a, I) \in [0, 1]$, and $\sum_{a \in A(I)} b_i(a, I) = 1$

  - $A(I)$ all actions available at information set I for i
  - (a, I) specific action a available in information set I
  - $b_i(a, I)$ probability attached to (a, I)

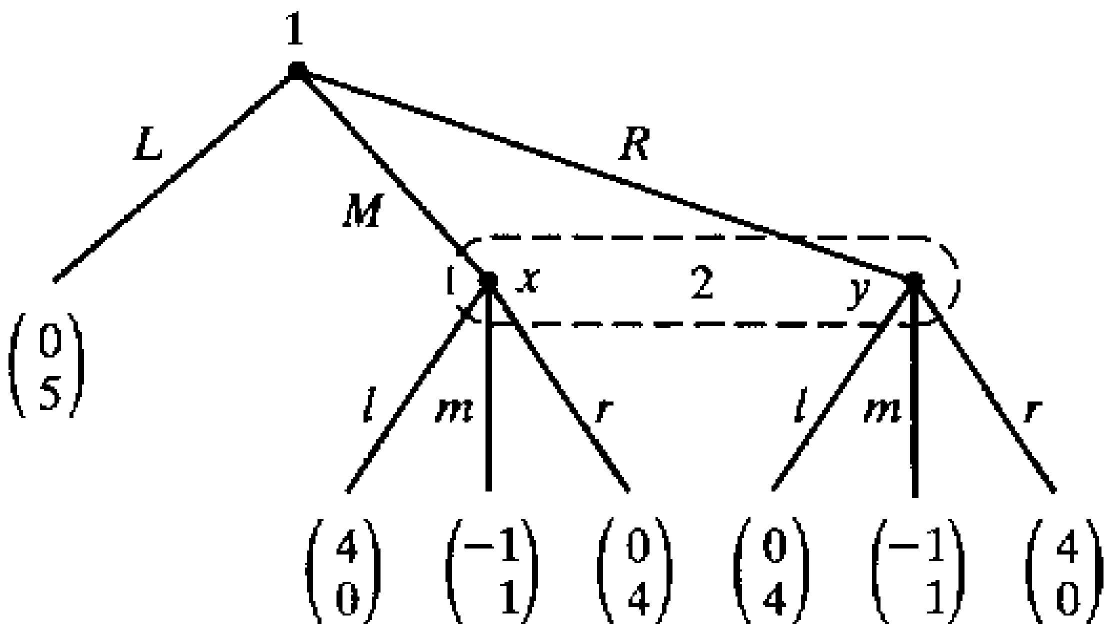


- $b_1(a, I_1) = (1/2, 1/2),\ b_1(a, I_2) = (2/3, 1/3)$
- *mixed* strategy $\leftrightarrow$ *behavioral* strategy
  - joint behavioral strategy: $b = (b_1, b_2, ..., b_N)$

Subgame perfection → reasonable strategies?



		Player 2		
		l	m	r
Player 1	L	0 , 5	0 , 5	0 , 5
	M	4 , 0	-1 , 1	0 , 4
	R	0 , 4	-1 , 1	4 , 0



SPNE: $\hat{s} = (L, m)$

- 2 never plays m, as for any belief about player 1 $p(M), p(R)$

  $s_2 = m$ yields lower expected payoff $(= 1)$ than $m_2 = (1/2, 0, 1/2)$

$\rightarrow$ SPNE nonsensical

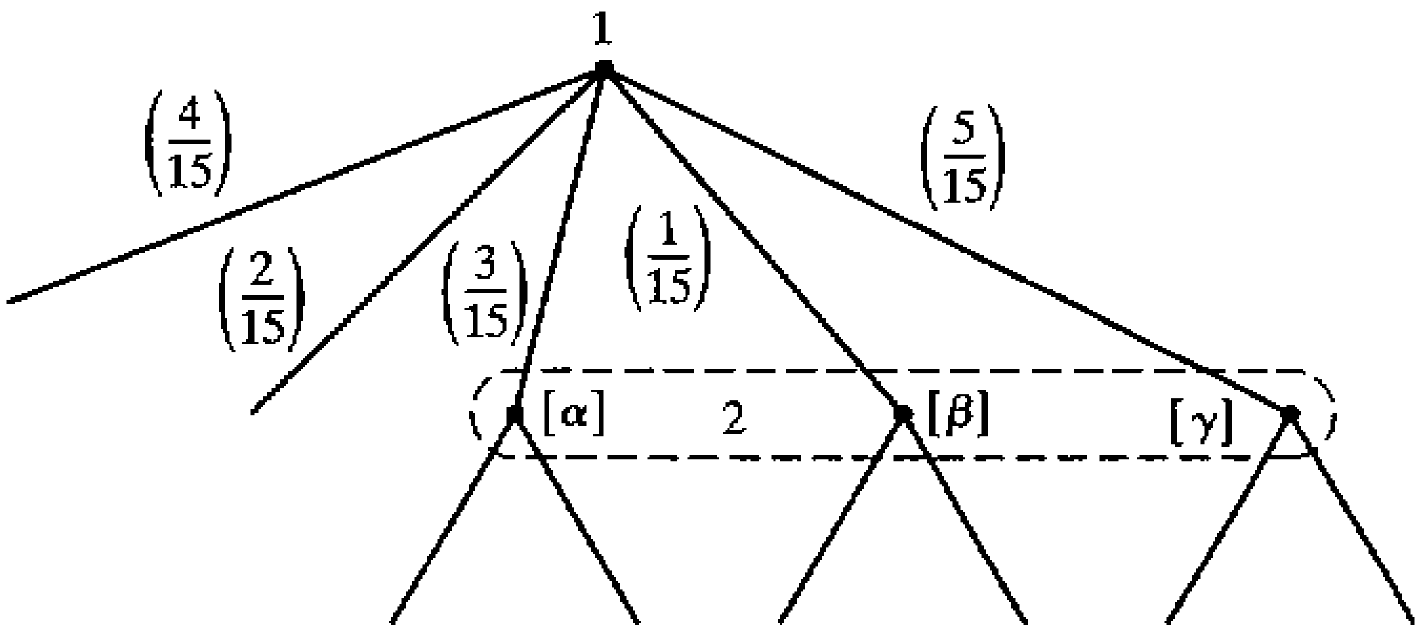
System of beliefs for given behavioral strategies

- Belief $p(x)$:

  at which node in $I(y)$ is player? $\sum_{x \in I(y)} p(x) = 1$

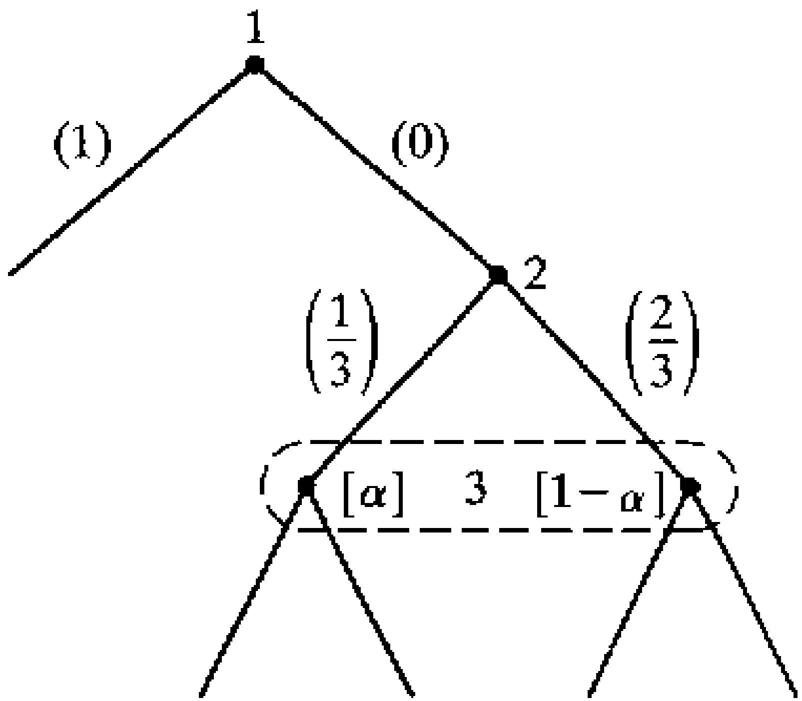
- System of beliefs p
- For given b, which p are sensible?
  - Bayes' rule, if possible
  - consistent assessment

- Bayes' rule: $p(x) = \frac{Pr(x|b)}{\sum_{y \in I} Pr(y|b)}$



beliefs: $[\alpha] = 1/3$, $[\beta] = 1/9$, $[\gamma] = 5/9$

- Bayes' rule is not always applicable



→ every belief satisfies Bayes' rule

Consistent assessment

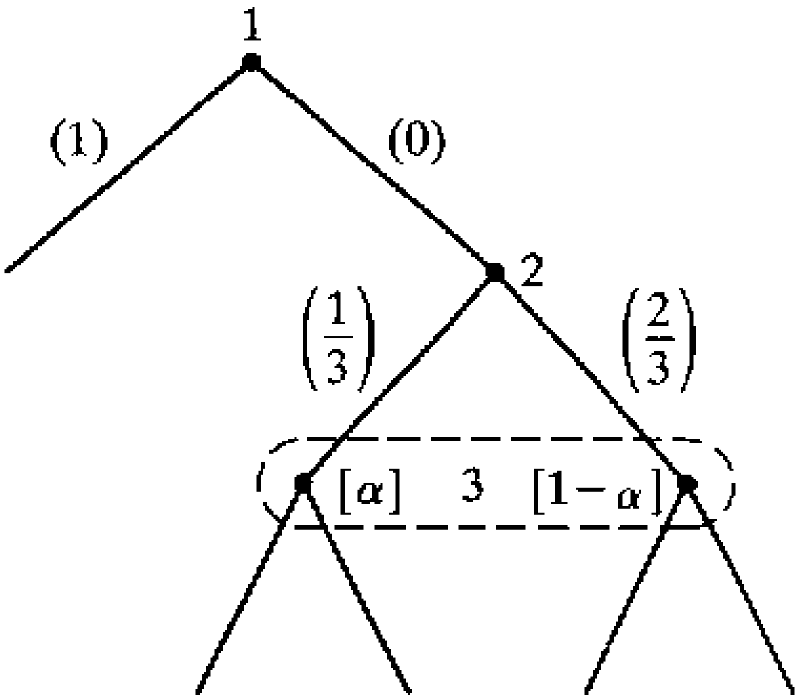
- Assessment: (b, p)
  - (b, p) is *sensible* if it satisfies consistent assessment

completely mixed behavioral strategy: $b_i(a, I) > 0$ for all i, a, I

- Bayes' rule can be applied

Definition. (b, p) is a consistent assessment if there exists a sequence of **completely mixed** behavioral strategies $b^n \to b$ such that associated Bayes' rule induced systems of beliefs $p^n \to p$.

- consistency requires what holds at all p^n to hold in limit p



- $b_1^n = (1 - 1/n, 1/n) \to (1, 0) = b$

  for all b_1^n, player 3's belief $p^n(\alpha) = 1/3$ by Bayes' rule

  $p^n(\alpha|I) = [(1/n)(1/3)]/(1/n) = 1/3$

  $p^n \to 1/3 = p.$

  consistency requires: for $b = (1, 0)$, $p(\alpha) = 1/3$

Sequential rationality

so far: how to form p for given b

2 steps left:

- **(a)** for a given p: what is best b (NE) : sequential rationality
- (b) combine both: sequential equilibrium

(i) fix assessment (p, b); calculate expected payoffs of I

$$v_i(p, b|I) = \sum_{x \in I} p(x)\, u_i(b|x)$$

(ii) compare $v_i(p, b|I)$ across b_i; find best responses

Definition. Assessment (p, b) is sequentially rational if $\forall i$, for **every** I_i, and every b_i':

$$v_i(p, b|I) \geq v_i(p, (b_i', b_{-i})|I)$$

Notes:

- assessment (p, b) need not be consistent
- sequential rationality captures every I_i, while SPNE only captures those I_i that are actually "reached"

Example for $v_1(p, b|I)$ for specific (p, b)

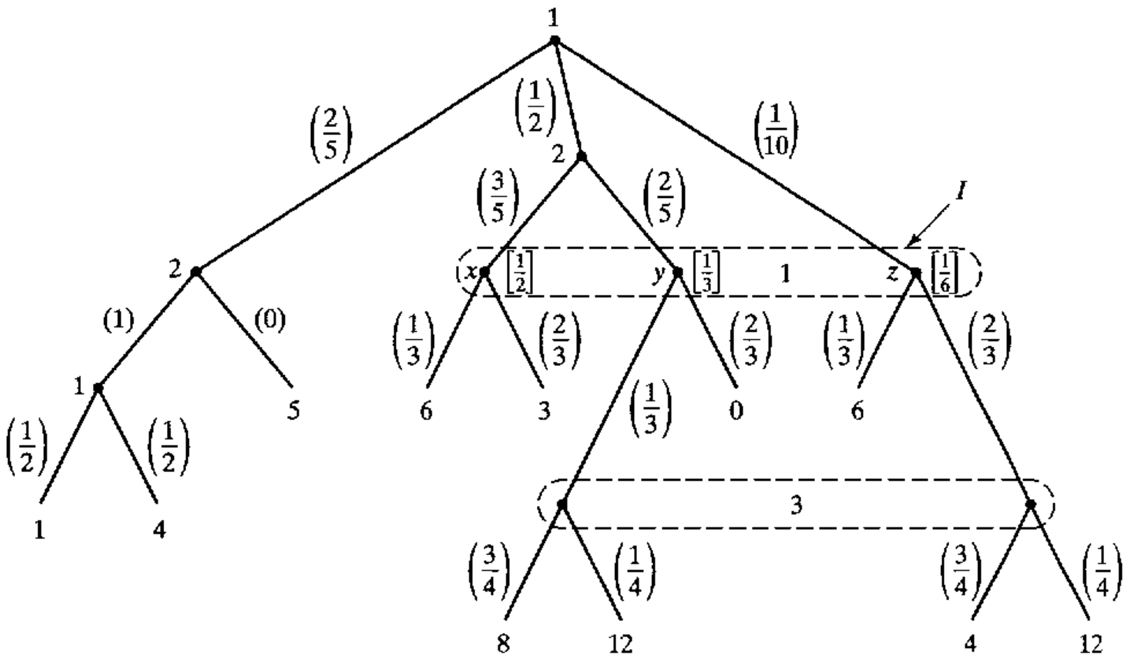


Figure 7.32. Payoffs conditional on an information set. See Fig. 7.33 for the calculation of 1's payoff conditional on I having been reached.

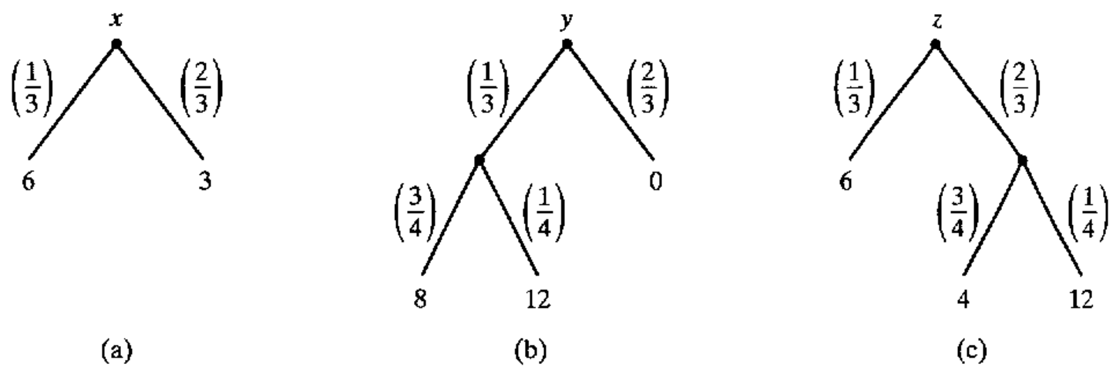
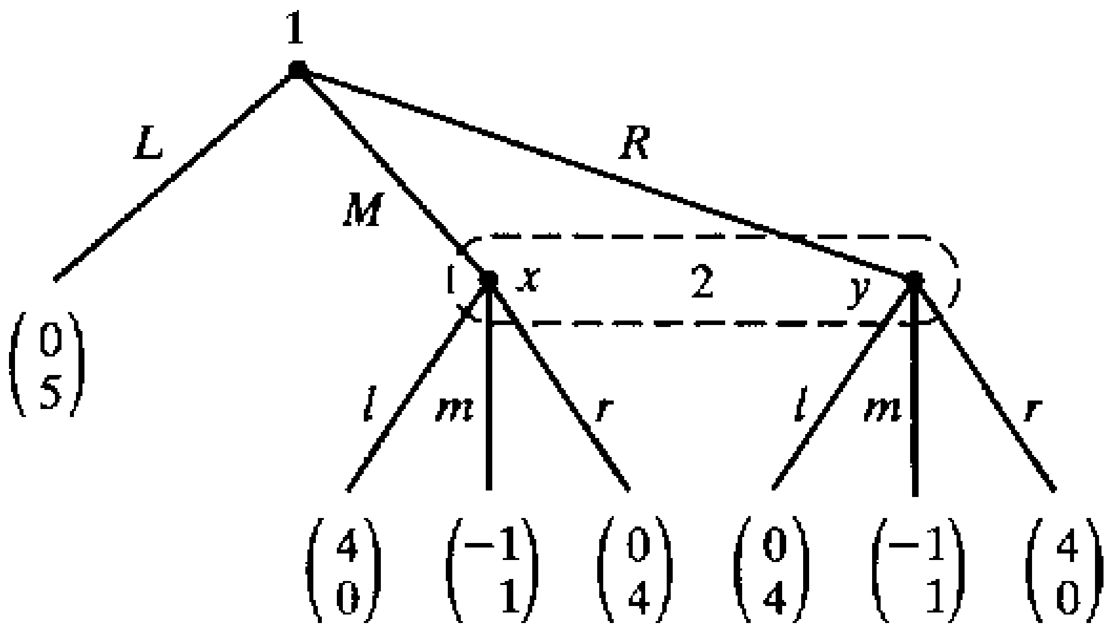


Figure 7.33. Calculating payoffs at an information set. Treating separately each node, x, y, and z within 1's information set labelled I in Fig. 7.32, we see from (a) that $u_1(b \mid x) = \frac{1}{3}(6) + \frac{2}{3}(3) = 4$, from (b) that $u_1(b \mid y) = \frac{1}{3}[\frac{3}{4}(8) + \frac{1}{4}(12)] + \frac{2}{3}[0] = 3$, and from (c) that $u_1(b \mid z) = \frac{1}{3}[6] + \frac{2}{3}[\frac{3}{4}(4) + \frac{1}{4}(12)] = 6$. Hence, $v_1(p, b \mid I) = p(x)u_1(b \mid x) + p(y)u_1(b \mid y) + p(z)u_1(b \mid z) = \frac{1}{2}(4) + \frac{1}{3}(3) + \frac{1}{6}(6) = 4$.

The nonsensical SPNE is not sequentially rational

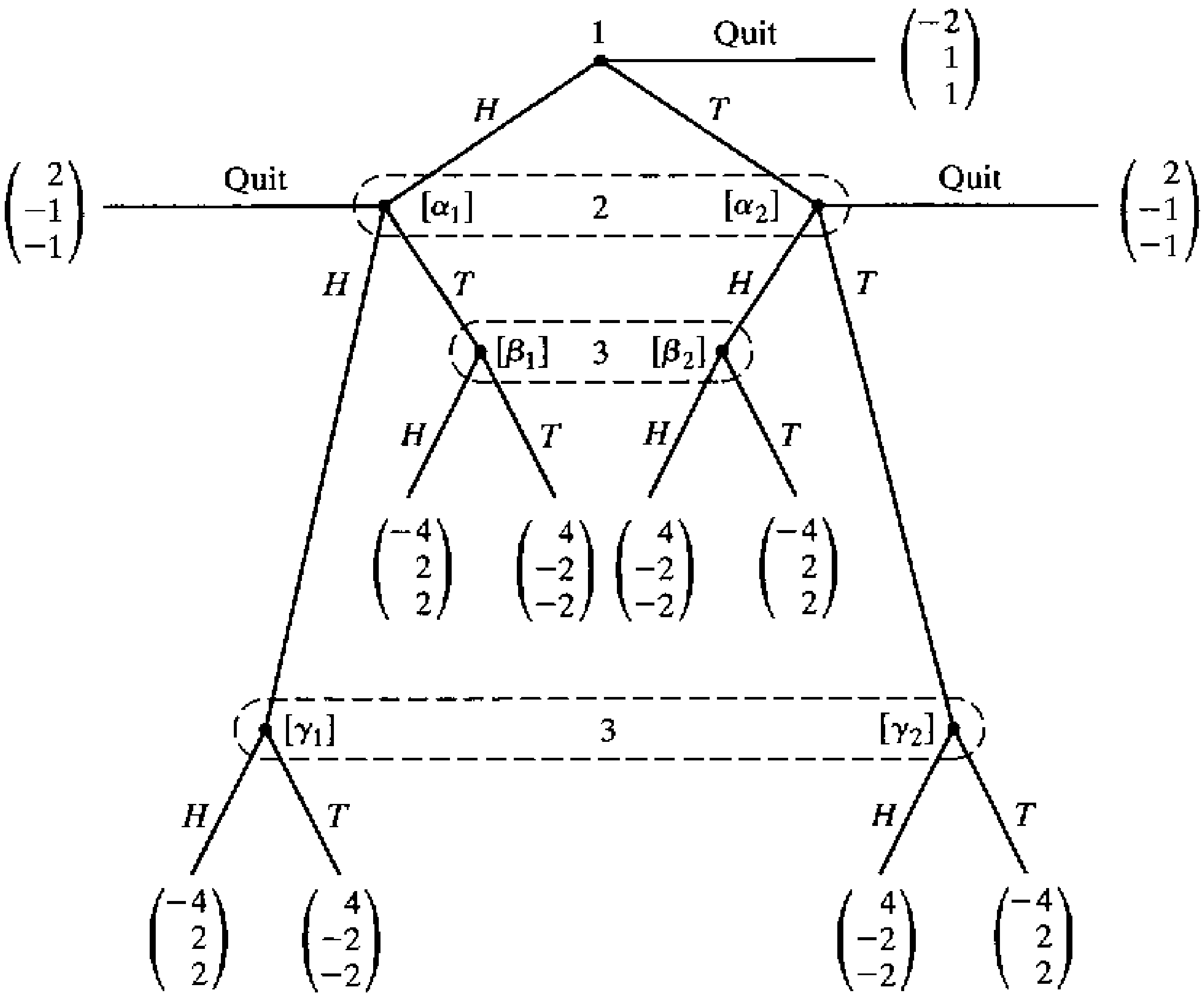


- Sequential rationality rules out $\hat{s} = (L, m)$
  - in I_2, sequential rationality is not consistent with player 2 choosing m
  - $v_2(p, (0, 1, 0)|I) < v_2(p, (1/2, 0, 1/2)|I)$
  - problem of SPNE was that I_2 is not reached during game

Sequential equilibrium (SE)

Definition.
Assessment (p, b) is a sequential equilibrium, if it satisfies both, consistent assessment and sequential rationality.

- Note. Every finite extensive form game (with perfect recall) possesses at least one sequential equilibrium. If (p, b) is a sequential equilibrium, then b is a SPNE.



- $(\alpha_1, \beta_1, \gamma_1, x, y, z_\beta, z_\gamma) = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$ is a SE, where x, y, and (z_β, z_γ) respectively are the behavioral strategies 1, 2, and 3 place on H on their respective information sets.

A SE in which no player quits...

assessment: $(p,b) = (\underbrace{\alpha_1, \beta_1, \gamma_1}_{p}, \underbrace{x, y, z_\beta, z_\gamma}_{b})$

- beliefs (Bayes): $\alpha_1 = x$, $\beta_1 = x\bar{y}/(x\bar{y} + y\bar{x})$, $\gamma_1 = xy/(xy + \bar{x}\bar{x})$
- sequential rationality:

$$v_1(H|I_1) = v_1(T|I_1)$$

$$v_2(H|I_2) = v_2(T|I_2)$$

$$v_3(H|I_{3\beta}) = v_3(T|I_{3\beta})$$

$$v_3(H|I_{3\gamma}) = v_3(T|I_{3\gamma})$$

Solving for a SE...

$$v_3(H|I_{3\gamma}) = \gamma_1\, 2 + \gamma_2\,(-2) = 4\gamma_1 - 2$$
$$v_3(T|I_{3\gamma}) = \gamma_1\,(-2) + \gamma_2\,(2) = -4\gamma_1 + 2$$

$$v_3(H|I_{3\beta}) = \beta_1\, 2 + \beta_2\,(-2) = 4\beta_1 - 2$$
$$v_3(T|I_{3\beta}) = \beta_1\,(-2) + \beta_2\,(2) = -4\beta_1 + 2$$

$$v_2(H|I_2) = \alpha_1\,[2z_\gamma - 2(1 - z_\gamma)] + \alpha_2\,[-2z_\beta + 2(1 - z_\beta)]$$
$$v_2(T|I_2) = \alpha_1\,[2z_\beta - 2(1 - z_\beta)] + \alpha_2\,[-2z_\gamma + 2(1 - z_\gamma)]$$

$$v_1(H|I_1) = y\,[-4z_\gamma + 4(1 - z_\gamma)] + \bar{y}\,[-4z_\beta + 4(1 - z_\beta)]$$
$$v_1(T|I_1) = y\,[4z_\beta - 4(1 - z_\beta)] + \bar{y}\,[4z_\gamma - 4(1 - z_\gamma)]$$

Note. $\alpha_2 = (1 - \alpha_1)$, $\beta_2 = (1 - \beta_1)$, $\gamma_2 = (1 - \gamma_1)$, $\bar{y} = 1 - y$

just enough equations to simultaneously determine (p, b)

$$(p, b) = (\alpha_1, \beta_1, \gamma_1, x, y, z_\beta, z_\gamma) = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$$