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Growth with Consumer Optimization

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The Plan

Monday:	An introduction to optimization in dynamic models Optimal control theory; Examples
Tuesday:	Application I: The Ramsey model; Exercises
Wednesday:	Application II: The AK model; Extensions and exercises

Today's Plan

- Motivating example
- Ingredients of static optimization problems
- Ingredients of an optimization problem in dynamic models
- Hamiltonian and Maximum principle (necessary conditions)
- Sufficient conditions
- Examples and exercises

Literature: Chiang, A.C. (1992), *Elements of dynamic optimization*, New York et al.: McGraw-Hill.

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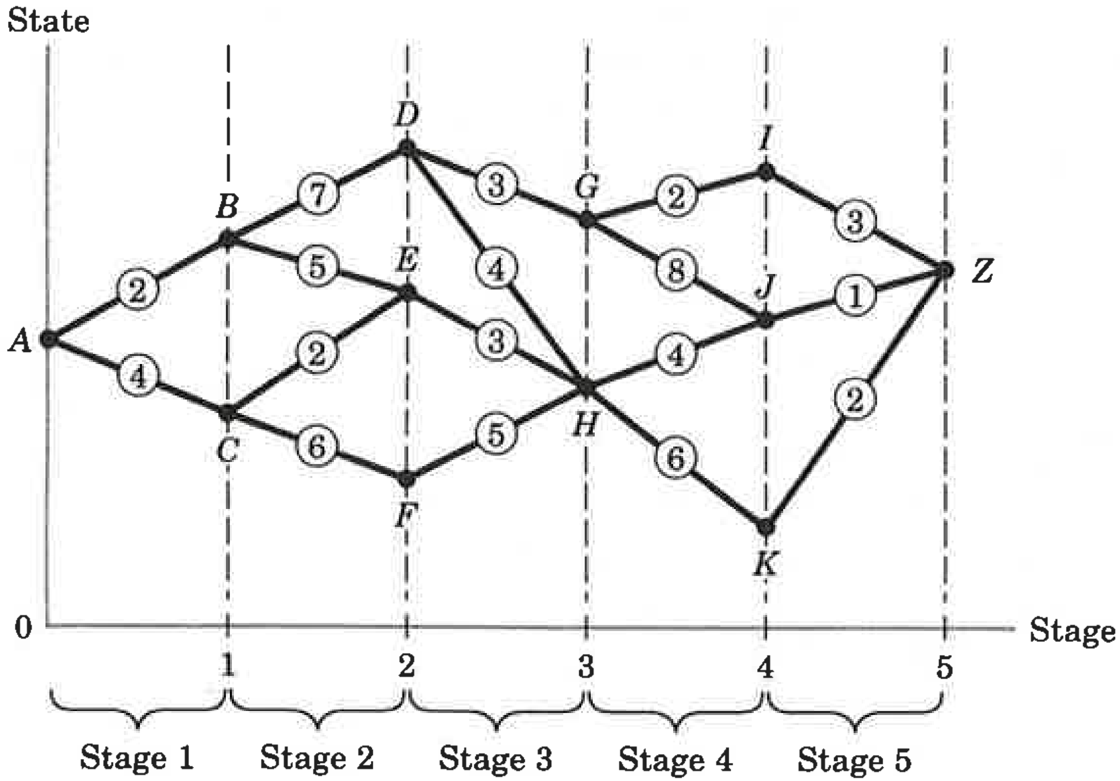
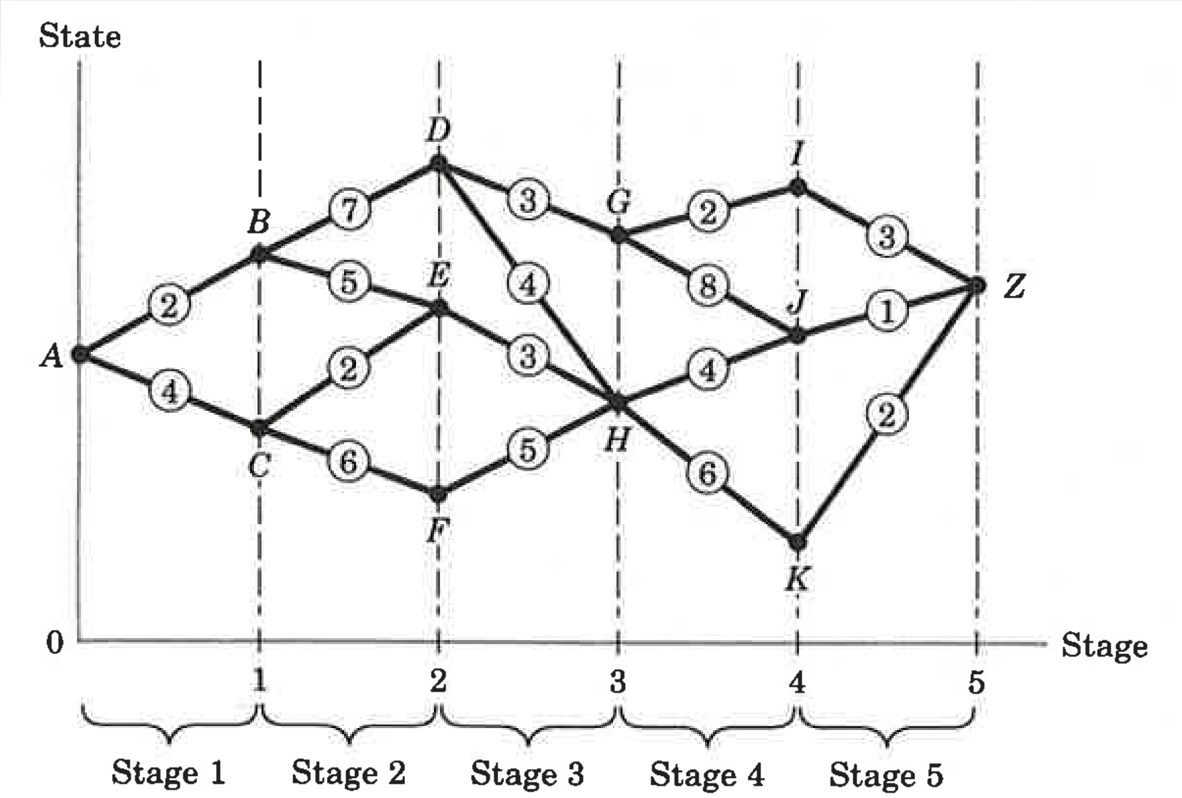
Motivating Discrete-Time Example: Cost Minimization



dynamic model: multistage decision making over time
available actions at A: $\{B, C\}$, etc.

cost (numbers in circles):
Which one is the cost-minimizing path?



Strategy 1: Starting at A, at each stage, choose the cost-min action at that stage: $\{B, E, H, J, Z\}$

Problem: Strategy 1 dominated by Strategy 2: $\{C, E, H, J, Z\}$

cost of Strategy 1 = 15
cost of Strategy 2 = 14 $\Rightarrow$ Strategy 1 is not optimal

Lesson: A myopic, one-stage-at-a-time optimization procedure does not, in general, yield the optimal path!

3 systematic approaches

- Dynamic programming
- Calculus of variation
- Optimal control theory (Maximum principle)

We will deal with optimal control theory

- continuous time
- infinite horizon problems
- autonomous problems
  (standard growth models)
- 1 choice variable and 1 state variable

Ingredients of Static Optimization

- Classical programming: ingredients

  choice variables $x = (x_1, x_2, x_3, ... x_N)$
  objective function $f(x)$
  equality constraint $0 = g(x)$

  - problem: $\max_x f(x)$ s.t. $0 = g(x)$
  - solution technique: Lagrange function
    $\mathcal{L}(x, \mu) = f(x) + \mu\, g(x)$
  - interior solution satisfies necessary first-order conditions (FOC)

$$\frac{\partial \mathcal{L}(x, \mu)}{\partial x_i} = 0\,, \quad \frac{\partial \mathcal{L}(x, \mu)}{\partial \mu} = 0\,, \quad i = 1, ..., N$$

Query. Identify these ingredients in the standard utility maximization problem.

Optimal control problem has same ingredients + a few others:

- variables are dated
- flow variables and stock variables
- constraint becomes dynamic (links variables across time)
- inital- and terminal conditions for stock variables

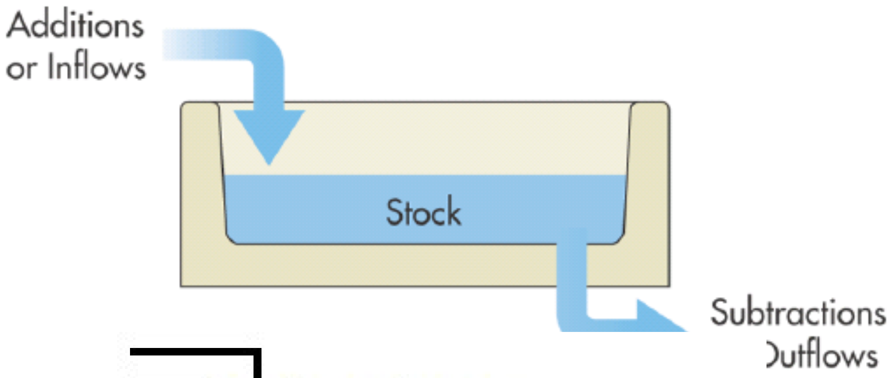
Digression: Flow variables versus stock variables

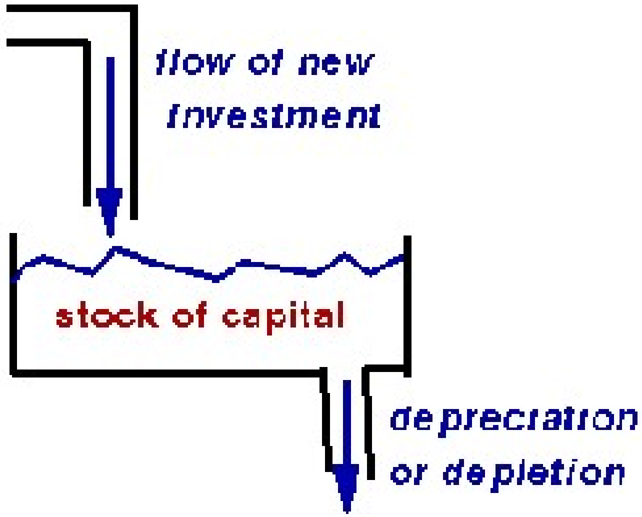
if time is considered continuous, a sharp distinction b/w stocks and flows is necessary

- stock: measured as a quantity at a given point in time (e.g., K)
- flow: measured as a quantity per unit of time (e.g., income, savings, consumption, investment)

- one cannot simply add stock and flow variables, as their units differ!

Two examples





Query. Which of the following variables are stocks/flows?

savings, consumption, income, wealth, labor input, your assets, production

- Stocks and flows in optimal control theory

  choice variables *are* flows (c, s, I, use of natural resources...)

  choice variables impact on stocks (wealth, capital, stock of natural resources,...)

  - impact described by *dynamic* constraint

    state variable changes in choice for control variable
    (think of bathtub example)

    today's consumption has an impact on wealth, thereby on tomorrows consumption possibilities

  State variables make the optimization problem a dynamic problem!

  Without state variables, a myopic, one-stage-at-a-time optimization procedure is optimal.

Ingredients in Optimal Control Problems

- Ingredients (specialized version)
  - two types of (dated) variables:
    - choice variables x_t "control variables"

      quantity per period of time;
      time period needed to give variable a meaning
    - stock variables y_t

      quantity at a point in time (not over period of time)
  - objective function $\int_0^\infty f(x_t, y_t)\, e^{-\rho\, t}\, dt$
  - dynamic constraint $\dot{y}_t = g(x_t, y_t)$ where $\dot{y}_t = \frac{\partial y_t}{\partial t}$
  - terminal condition "transversality condition"

    What happens to y_t (or its value) as $t \to \infty$?

    $\to$ required to pick best path of $(x_t)_{t=0}^{\infty}$

Ingredients cont.'ed

Economic growth models: let $y_t = k_t^\alpha$ (*)

- *Query.* Suppose $Y_t = K_t^\alpha L_t^{1-\alpha}$, and $y_t \equiv Y_t/L_t$, $k_t \equiv K_t/L_t$. Derive (*)!
- typical dated variables

  typical control (flow) variable: c_t
  typical stock variable k_t
- typical (intertemporal) objective function

  $\int_0^\infty u(c_t) e^{-\rho\, t}\, dt$
- typical dynamic constraint

  $\dot{k}_t = g(c_t, k_t) = y_t - c_t - \delta\, k_t = k_t^\alpha - c_t - \delta\, k_t$
- terminal = transversality condition

  whatever stock of capital is left over at the end of time, $t \to \infty$, must have no value according to the intertemporal objective function

Maximum Principle

- Problem: choose $(x_t)_{t=0}^{\infty}$ so as to max objective function

  s.t.

  $x_t \geq 0$

  $\dot{y}_t = g(x_t, y_t, t),\ y_0$ given

  transversality condition

- Solution technique: *current value* Hamiltonian function

  $\mathcal{H}(x_t, y_t, \mu_t) = f(x_t, y_t) + \mu_t\, g(x_t, y_t)$

- Maximum Principle: At an interior solution, there exists a continuous μ_t so that $(x_t, y_t)_{t=0}^{\infty}$ satisfies first-order conditions

$$\frac{\partial \mathcal{H}(.)}{\partial x_t} = 0\,, \quad \frac{\partial \mathcal{H}(.)}{\partial \mu_t} = \dot{y}_t\,, \quad \frac{\partial \mathcal{H}(.)}{\partial y_t} = \rho \mu_t - \dot{y}_t$$

  and the transversality condition $\lim_{t\to\infty} \mu_t\, y_t\, e^{-\rho\, t} = 0$.

Hamiltonian

- Hamiltonian $\leftrightarrow$ Lagrangian
- μ_t = shadow price of y_t in objective function

Necessary FOC

- $\frac{\partial \mathcal{H}(.)}{\partial x_t} = 0$
  marginal benefit takes into account the impact of x_t on $\dot{y}_t$
- $\frac{\partial \mathcal{H}(.)}{\partial \mu_t} = \dot{y}_t$
  ensures dynamic constraint
- $\frac{\partial \mathcal{H}(.)}{\partial y_t} = \rho \mu_t - \dot{y}_t$
  marginal benefit of an additional unit of y_t "consumed" today on objective function

Transversality condition (TC)

- $\lim_{t \to \infty} \mu_t \, y_t \, e^{-\rho \, t} = 0$

  Query. Suppose y_t is some household's capital stock, and ρ is a discount rate. Give an intuitive interpretation of the transversality condition.

- TC determines the initial value of the control variable

  (the initial value of the state variable is exogenously given)

Second Order Conditions

Maximum Principle provides necessary FOC

Sufficient conditions (Arrow)

- Consider $H^0(y, \mu) \equiv f(x^*, y) + \mu\, g(x^*, y)$

  If, for a given μ, $H^0(y, \mu)$ is concave in y for all t, the above conditions are also sufficient (not only necessary) for a global maximum.

Examples

A few hints for the examples

- notation: x_t means x as a function of time t in continuous time
- $\dot{x}_t = \frac{\partial x_t}{x_t}$
- growth rate $\frac{\dot{x}_t}{x_t}$
- simple differential equation

$$\dot{y}_t + a y_t = b_t \,, \quad y_0 \text{ given} \,, \quad t \geq 0$$

$$y_t = y_0 \, e^{-a\,t} + \int_0^t b_\tau \, e^{-a(t-\tau)} \, d\tau$$

Example 1

Choose $(x_t)_{t=0}^{\infty}$ so as to

$$\max \int_0^{\infty} (1 - x_t^2) e^{-\rho t}$$

$$\text{s.t. } x_t \geq 0 \quad \text{(control region)}$$

$$\dot{y}_t = x_t \,, \quad y_0 > 0 \text{ given}$$

Set up the optimal control problem and solve for $(x_t, y_t, \mu_t)_{t=0}^{\infty}$.

Remark. This is a degenerate problem. x impacts on y but y neither restricts (future) choices of x, nor does y show up in the objective function. So, y has no value, and we expect its shadow price $\mu_t = 0$.

Example 2 (Optimal oil extraction)

A firm extracts oil from an oil field.

y_t stock of oil in the oil field (number of barrels)
x_t rate of extraction per unit of time (barrels of oil)
q_t market price per barrel of oil
$c(x_t, y_t)$ cost of extraction per unit of time

The firm wants to extract oil so as to max present value of profits, $\pi(x_t, y_t)$, given the constant discount rate ρ.

(a) Set up the optimal control problem.

(b) Suppose (i) $\dot{q}_t = 0$, (ii) $c(x_t, y_t) = \alpha\, x_t$. Calculate the optimal growth rate of extraction. Is it growing or declining over time? Why?

Example 3

Consider $\rho = 0$. Choose $(x_t)_{t=0}^{1}$ so as to

$$\max \int_0^1 -x_t^2 \, dt$$

$$\text{s.t. } \dot{y}_t = y_t + x_t$$

$$y_0 = 1 \, , \; y_1 = 0$$

Set up the optimal control problem and calculate the optimal $(x_t)_{t=0}^{1}$. Show that the sufficient SOC are satisfied.