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Application I. Ramsey Model

Ronald Wendner

Department of Economics
University of Graz, Austria

The Ramsey – Cass – Koopmans Model

- Recap of optimal control theory
- Market conditions
- Agents (households, firms)
- Dynamic system
- Steady state (BGP), stability
- Saving rate
- Examples

Recap: Optimal control

- Main ingredients
 - control- and state variables (x_t, y_t)
 - instantaneous (date- t -) objective function $f(x_t, y_t)$
 - intertemporal objective function $\int_0^{\infty} f(x_t, y_t) e^{-\rho t}$
 - dynamic constraint $\dot{y}_t = g(x_t, y_t)$
 - initial- and end-of-time- (=transversality) conditions for the *state* variable

- Problem

- $(x_t, y_t)_{t=0}^{\infty}$ is admissible if dynamic- and boundary constraints are satisfied
- among all admissible paths for $(x_t, y_t)_{t=0}^{\infty}$ find the one(s) that maximize the intertemporal objective function $\int_0^{\infty} f(x_t, y_t) e^{-\rho t}$
- such a pair $(x_t, y_t)_{t=0}^{\infty}$ is called *optimal* or a solution to the optimal control problem
- for finding the solution, we employ the Hamiltonian function

- Procedure

1. Set up the (current value) Hamiltonian

$$H(x_t, y_t, \mu_t) = f(x_t, y_t) + \mu_t g(x_t, y_t)$$

2. At every point in time, maximize $H(\cdot)$ with respect to the control variable: $\frac{\partial H(\cdot)}{\partial x_t} = 0$

3. Differentiate $H(\cdot)$ w.r.t. the state variable and set the result equal to $(\rho\mu_t - \dot{\mu}_t)$. That is, $\frac{\partial H(\cdot)}{\partial y_t} = (\rho\mu_t - \dot{\mu}_t)$ for all t .

4. Now, use the *Maximum Principle*. It says that for an interior solution, $(x_t, y_t)_{t=0}^{\infty}$, there exists a continuous function μ_t , such that for all t (2.) and (3.) hold, *and* the transversality condition $\lim_{t \rightarrow \infty} \mu_t y_t e^{-\rho t} = 0$ is satisfied.

Market Conditions

- Markets

 - output (“all-purpose” output good)

 - labor

 - capital services (rental market for capital goods)

- Agents

 - 1 representative household (family) with $L_t = L_0 e^{nt} = e^{nt}$ members (“consumers”)

 - firms (technology) as in Solow model: CD technology

- 1 Asset: ownership claims on capital (K)

- No uncertainty

- Perfect competition

 - prices exogenous to agents

 - real prices are measured in terms of the output price

 - CRS technology: no profits

Notation alert

- uppercase letters: aggregate variables (K_t, Y_t, C_t)
- lowercase letters: per capita variables
($k_t = K_t/L_t, y_t = Y_t/L_t, c_t = C_t/L_t$)
- level of technology A_t (index, number of patents)
technical progress = growth rate of A_t , that is: \dot{A}_t/A_t
- effective labor input: $A_t L_t$
- lowercase-tilde-letters: per **effective labor** variables

$$\tilde{k}_t = K_t / (L_t A_t)$$

$$\tilde{y}_t = Y_t / (L_t A_t)$$

$$\tilde{c}_t = C_t / (L_t A_t)$$

Households

- Asset: capital $K_t = k_t L_t = \tilde{k}_t (A_t L_t)$
- Representative household belongs to family with L_t members

$$U_0 = \int_0^{\infty} (\ln c_t) L_t e^{-\rho t} dt = \int_0^{\infty} (\ln c_t) e^{-(\rho-n)t} dt$$

ρ *pure* rate of time preference
 $(\rho - n)$ *effective* rate of time preference

- Flow budget constraint

$$\dot{K}_t = r_t K_t + w_t L_t - c_t L_t, \quad K_0 \text{ given}$$

$$\dot{k}_t = (r_t - n)k_t + w_t - c_t, \quad k_0 \text{ given}$$

- Solvency (No-Ponzi-Game) condition (NPG)

$$\lim_{t \rightarrow \infty} K_t e^{-\int_0^t r_s ds} \geq 0$$

use $K_t = k_t L_t = k_t e^{n t}$, then:

$$\lim_{t \rightarrow \infty} k_t e^{-\int_0^t (r_s - n) ds} \geq 0$$

$r_t - n =$ growth-corrected rate of interest

- Representative consumer's problem:

choose $(c_t)_{t=0}^{\infty}$ so as to maximize U_0

subject to:

- (i) $c_t \geq 0$
- (ii) flow budget constraint
- (iii) solvency condition

Optimization part

- Optimal control theory in the Ramsey model

- two types of variables:

- control variable c_t
- stock variables k_t

- objective function $\int_0^{\infty} (\ln c_t) e^{-(\rho-n)t} dt$

- dynamic constraint $\dot{k}_t = (r_t - n)k_t + w_t - c_t$

- transversality condition (TVC)

$$\lim_{t \rightarrow \infty} \mu_t k_t e^{-(\rho-n)t} = 0$$

Solution procedure

- 1 set up *current-value* Hamiltonian

$$H(c_t, k_t, \mu_t) = (\ln c_t) + \mu_t[(r_t - n)k_t + w_t - c_t]$$

- 2 choose c_t so as to max Hamiltonian

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0 \quad (1)$$

- 3 differentiate Hamiltonian w.r.t state variable

$$\frac{\partial H}{\partial k_t} = (r_t - n)\mu_t = (\rho - n)\mu_t - \dot{\mu}_t \Leftrightarrow -\frac{\dot{\mu}_t}{\mu_t} = r_t - \rho \quad (2)$$

- 4 apply Maximum Principle
(for picking U_0 -max consumption-/saving path)

$$\lim_{t \rightarrow \infty} \mu_t k_t e^{-(\rho - n)t} = 0 \quad (3)$$

$$(1) \frac{1}{c_t} = \mu_t \quad \text{MB} = \text{MC}$$

$$(2) r_t = \rho - \frac{\dot{\mu}_t}{\mu_t} \quad \text{return to savings} = \text{required return}$$

$$(3) \lim_{t \rightarrow \infty} \mu_t k_t e^{-(\rho-n)t} = 0$$

< violates NPG

> k with positive contribution to utility “left over”

Remark: FOC (2) + TVC (3) imply NPG condition

▶ Proof

Ramsey

- Growth of per capita consumption

differentiate (1) w.r.t. time & divide both sides by μ_t

$$c_t^{-1} = \mu_t \Rightarrow -\frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho) \quad \text{by (2)}$$

$$\frac{\dot{c}_t}{c_t} = r_t - \rho$$

→ Keynes-Ramsey rule

Firms

- Setup (\rightarrow Solow model)

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

$$\dot{A}_t/A_t = \gamma \geq 0$$

$$\dot{L}_t/L_t = n \geq 0$$

per unit of effective labor (AL)

$$\tilde{y}_t = \tilde{k}_t^\alpha$$

- Factor markets

$$\hat{r}_t = r_t + \delta = \partial Y_t / (\partial K_t) = \alpha \tilde{y}_t / \tilde{k}_t \Rightarrow r_t = \alpha \tilde{y}_t / \tilde{k}_t - \delta$$

$$w_t/A_t = \partial Y_t / (\partial L_t) = (1 - \alpha) \tilde{y}_t$$

- Capital accumulation

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

Dynamic system

- Two “dynamic variables”: capital \tilde{k}_t , consumption \tilde{c}_t

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

$$\Leftrightarrow \frac{\dot{K}_t}{K_t} \frac{K_t}{A_t L_t} = \frac{Y_t}{A_t L_t} - \frac{C_t}{A_t L_t} - \delta \frac{K_t}{A_t L_t}$$

$$\Leftrightarrow \frac{\dot{K}_t}{K_t} \tilde{k}_t = \tilde{y}_t - \tilde{c}_t - \delta \tilde{k}_t$$

$$\text{as } \frac{\dot{\tilde{k}}_t}{\tilde{k}_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} - (\gamma + n)$$

$$\Rightarrow \dot{\tilde{k}}_t = \tilde{y}_t - \tilde{c}_t - (\gamma + \delta + n) \tilde{k}_t$$

$$\Leftrightarrow \dot{\tilde{k}}_t = \tilde{k}_t^\alpha - \tilde{c}_t - (\gamma + \delta + n) \tilde{k}_t$$

Dynamic System

- Consumption

$$\frac{\dot{c}_t}{c_t} = (r_t - \rho) = [(\alpha \tilde{k}_t^{\alpha-1} - \delta) - \rho]$$

as $\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{\dot{c}_t}{c_t} - \gamma$

$$\Rightarrow \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = [(\alpha \tilde{k}_t^{\alpha-1} - \delta) - \rho - \gamma]$$

- Dynamic system in (\tilde{k}, \tilde{c})

$$\dot{\tilde{k}}_t = \tilde{k}_t^\alpha - \tilde{c}_t - (\gamma + \delta + n) \tilde{k}_t \quad (4)$$

$$\dot{\tilde{c}}_t = [(\alpha \tilde{k}_t^{\alpha-1} - \delta) - \rho - \gamma] \tilde{c}_t \quad (5)$$

$$\tilde{k}_0 \text{ given ; TVC} \quad (6)$$

Grafic Analysis of Transitional Dynamics

- Idea: investigate the $\dot{\tilde{k}} = 0, \dot{\tilde{c}} = 0$ -loci in (\tilde{k}, \tilde{c}) -plane

$\dot{\tilde{k}} = 0$ -locus: $\tilde{c} = \tilde{k}^\alpha - (\gamma + \delta + n)\tilde{k} \equiv \tilde{c}(\tilde{k})$

inversely U-shaped with maximum at \tilde{k}_{GR} :

$$f'(\tilde{k}_{GR}) - \delta = \gamma + n \quad \Rightarrow \quad \tilde{k}_{GR} = \left[\frac{\alpha}{\gamma + \delta + n} \right]^{1/(1-\alpha)}$$

points above $\tilde{c}(\tilde{k})$: $\dot{\tilde{k}} < 0$ (too much \tilde{c}); points below $\tilde{c}(\tilde{k})$: $\dot{\tilde{k}} > 0$

$\dot{\tilde{c}} = 0$ -locus satisfies for all \tilde{c} :

vertical line at \tilde{k}_{KR} (modified GR):

$$f'(\tilde{k}_{KR}) - \delta = \rho + \gamma \quad \Rightarrow \quad \tilde{k}_{KR} = \left[\frac{\alpha}{\rho + \gamma + \delta} \right]^{1/(1-\alpha)}$$

points to the right: $\dot{\tilde{c}} < 0$ (too low r); points to the left: $\dot{\tilde{c}} > 0$

Grafic Analysis of Transitional Dynamics

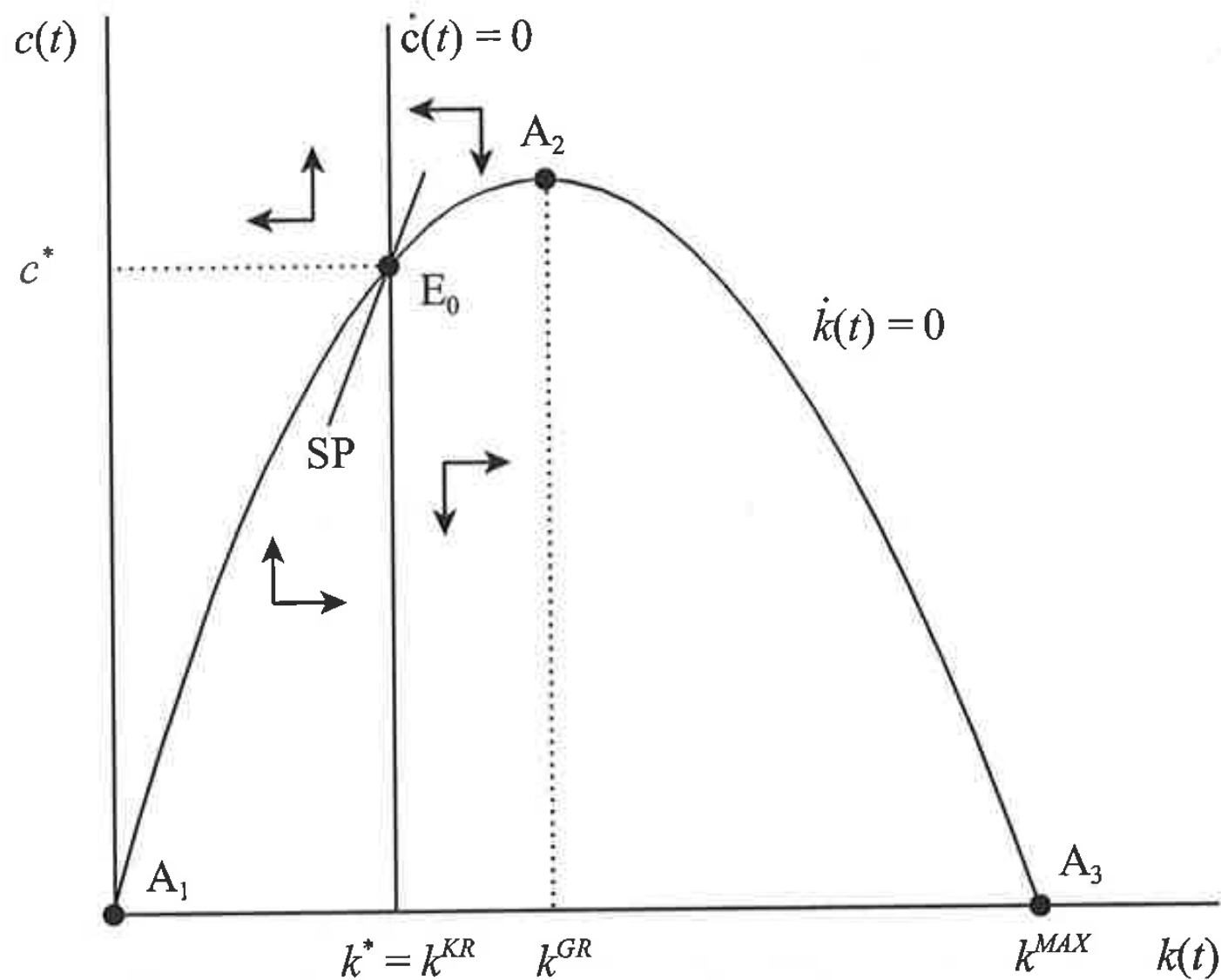


Figure 13.9: Phase diagram of the Ramsey model

Balanced Growth Path (steady state)

- BGP (steady state)
 - (k, y, c) grow at constant rates (stylized facts)
 - from capital accumulation $\dot{k}/k = y/k - c/k - (\delta + n)$
 $\Rightarrow g_k = g_y = g_c$
 - from production $g_k = g_y \Rightarrow g_k = g_y = \gamma$
 - $(\tilde{k}, \tilde{y}, \tilde{c})$ constant
- BGP characterized by intersection of $\dot{\tilde{k}} = 0, \dot{\tilde{c}} = 0$ -loci in (\tilde{k}, \tilde{c}) -plane
 E_0 in the [▶ figure](#)

Balanced Growth Path

- Existence

TVC must be satisfied $\Leftrightarrow \rho + \gamma > \gamma + n$

BGP $\tilde{k}_{KR} < \tilde{k}_{GR}$

- BGP

$$f'(\tilde{k}^*) - \delta = \rho + \gamma \Rightarrow$$

$$\tilde{k}^* = \left[\frac{\alpha}{\rho + \gamma + \delta} \right]^{1/(1-\alpha)}, \quad \tilde{y}^* = (\tilde{k}^*)^\alpha$$

$$\tilde{c}^* = (\tilde{k}^*)^\alpha - (\gamma + \delta + n)\tilde{k}^*$$

Ramsey

- per capita growth γ (exogenous)
- per capita income

saving rate

$$s^* = \frac{\tilde{y}^* - \tilde{c}^*}{\tilde{y}^*} = \frac{\alpha(\gamma + \delta + n)}{\delta + \rho + \gamma}$$

per capita income in terms of saving rate

$$y_t^* = \left[\frac{s^*}{n + \gamma + \delta} \right]^{\alpha/(1-\alpha)} A_t$$

→ “rich” if: A_t, s^* high; n low (→ Solow model)

→ theory of evolution and long-run level of s_t

s^* determined jointly by preference- and technology parameters

e.g., impatience (ρ)

Ramsey

Value added (w.r.t. Solow)

- endogenous saving rate
- household behavior responds to shocks (e.g., tax shocks)
- model opens up for welfare analysis
- today one workhorse model for analyzing economic policy

Examples

Example 1

In the basic setup of the Ramsey model, we used a logarithmic instantaneous utility function. If instead we employ a CIES- (constant intertemporal elasticity of substitution) utility function (see below), calculate the economy's growth rate, saving rate, and per capita income along a BGP. How does the intertemporal elasticity of substitution affect (g_c, s, y) ?

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}, \quad \theta > 0$$

Example 2.

Let us return to the logarithmic utility function. Suppose, the following tax program is introduced. The government taxes capital income at some constant rate, τ_k and rebates the tax revenue in a lump sum fashion back to all households. How does this “tax reform” affect the saving rate and per capita income along a BGP?

Ramsey

- Remark: FOC (2) + TVC (3) imply NPG condition

from (2), $\mu_t = \mu_0 e^{-\int_0^t r_s - \rho ds}$, $\mu_0 \neq 0$

into (3) + dividing by μ_0 yields

$$\lim_{t \rightarrow \infty} k_t e^{-\int_0^t (r_s - n) ds} = 0 \quad \Rightarrow \quad \lim_{t \rightarrow \infty} k_t e^{-\int_0^t (r_s - n) ds} \geq 0 \quad (\text{NPG})$$