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Application II. AK Model and Extensions

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(Fully) Endogenous Growth

- Recap of Ramsey model
- Households
- Firms
- Endogeneous growth (BGP)
- Examples

Recap of Ramsey model

- per capita income *growth*: γ exogenous
- per capita income *level*:

  $\rightarrow$ "rich" if: A_t, s^* high; n low ($\rightarrow$ Solow model)

  $\rightarrow$ theory of evolution and long-run level of s_t

  s^* determined jointly by preference- and technology parameters

Value added (w.r.t. Solow)

- endogenous saving rate
- household behavior responds to shocks (e.g., tax shocks)
- model opens up for welfare analysis

Households

- household side does not change

representative household belongs to family with L_t members

$$U_0 = \int_0^\infty (\ln c_t)\, L_t\, e^{-\rho t} dt = \int_0^\infty (\ln c_t)\, e^{-(\rho - n)t} dt$$

household budget (flow) constraint

$$\dot{k}_t = (r_t - n)k_t + w_t - c_t\,, \quad k_0 \text{ given}$$

transversality

$$\lim_{t\to\infty} \mu_t\, k_t\, e^{-(\rho-n)\,t} = 0$$

Technology

- simplest case: $y_t = Ak_t$
- factor prices

$$r_t = \frac{\partial y_t}{\partial k_t} - \delta = A - \delta$$

$$w_t = \frac{\partial y_t}{\partial L_t} = 0$$

- household budget constraint

$$\dot{k}_t = (A - \delta - n)k_t - c_t\,, \quad k_0 \text{ given}$$

Ingredients of optimal control problem

- intertemporal utility function

$$U_0 = \int_0^\infty (\ln c_t)\, e^{-(\rho-n)t} dt$$

household budget constraint

$$\dot{k}_t = (A - \delta - n)k_t - c_t\,, \quad k_0 \text{ given}$$

transversality

$$\lim_{t\to\infty} \mu_t\, k_t\, e^{-(\rho-n)\,t} = 0$$

- 1. Set up Hamiltonian function

$$H(c_t, k_t, \mu_t) = \ln c_t + \mu_t[(A - \delta - n)k_t - c_t]$$

- 2. Maximize the Hamiltonian w.r.t. control variable c_t

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0 \quad \Rightarrow \quad \mu_t = c_t^{-1} \quad \Rightarrow \quad \frac{\dot{\mu}_t}{\mu_t} = -\frac{\dot{c}_t}{c_t}$$

- 3. Differentiate H w.r.t. state variable k_t and set $=$ to $(\rho - n)\mu_t - \dot{\mu}_t$

$$\frac{\partial H}{\partial k_t} = \mu_t(A - \delta - n) = (\rho - n)\mu_t - \dot{\mu}_t \quad \Rightarrow \quad \frac{\dot{\mu}_t}{\mu_t} = -(A - \delta - \rho)$$

- Use 2. and 3. to calculate the consumption growth rate per-capita consumption growth rate

$$\frac{\dot{c}_t}{c_t} = (A - \delta - \rho)$$

per-capita capital growth rate: rewrite dynamic constraint

$$\frac{\dot{k}_t}{k_t} = (A - \delta - n) - \frac{c_t}{k_t}$$

- BGP requires $\frac{\dot{k}_t}{k_t} = \frac{\dot{c}_t}{c_t}$. Why?

$$\Rightarrow \frac{\dot{k}_t}{k_t} = (A - \delta - \rho)$$

per-capita income growth: as $y_t = Ak_t$:

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} = (A - \delta - \rho)$$

Parameter restrictions

- positivity of growth: $\rho < A - \delta$
- boundedness of utility integral: $\rho > n$

Endogenous growth: rises in A and declines in ρ

$\rightarrow$ via saving rate

$$s = \frac{y - c}{y} = \frac{y/k - c/k}{y/k} = \frac{A - (\rho - n)}{A}$$

- AK-model explains the saving rate *and* the growth rate of per capita income and -consumption.

Examples

Example 1

Suppose, the following tax program is introduced. The government taxes capital income at some constant rate, τ_k and rebates the tax revenue in a lump sum fashion back to all households. How does this "tax reform" affect (i) the saving rate, and (ii) the endogenous per capita income growth rate?

Example 2.

As before, let $\rho > n$ to ensure a bounded utility integral. How does the endogenous growth rate change, if households not only derive utility from consumption, but also from wealth, k_t in addition? Specifically, consider the following utility function:

$$u(c_t, k_t) = \ln c_t + s \ln k_t \,, \quad s \geq 0 \,.$$

Does the preference for wealth raise the endogenous growth rate?