KARL-FRANZENS-UNIVERSITÄT GRAZ
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Growth with Consumer Optimization

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The Plan

- Motivating example
- Ingredients of static optimization problems
- Ingredients of an optimization problem in dynamic models
- Hamiltonian and Maximum principle (necessary conditions)
- Sufficient conditions
- Examples and exercises

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Literature: Chiang, A.C. (1992), *Elements of dynamic optimization*, New York et al.: McGraw-Hill.

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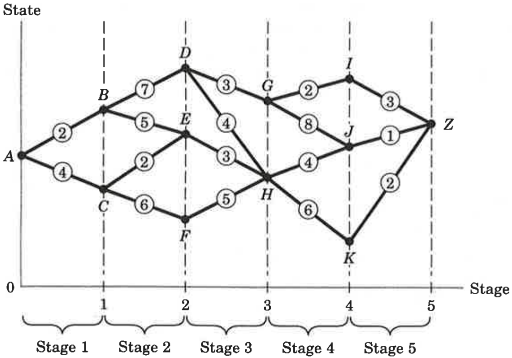
Motivating Discrete-Time Example: Cost Minimization



dynamic model: multistage decision making over time
available actions at A: $\{B, C\}$, etc.

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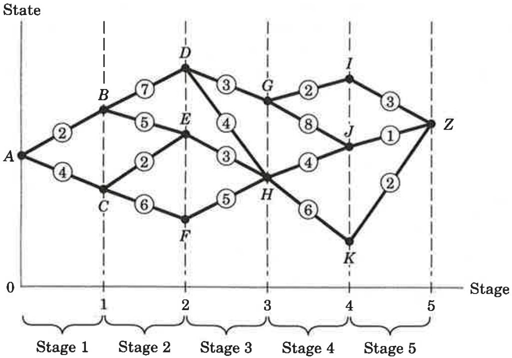
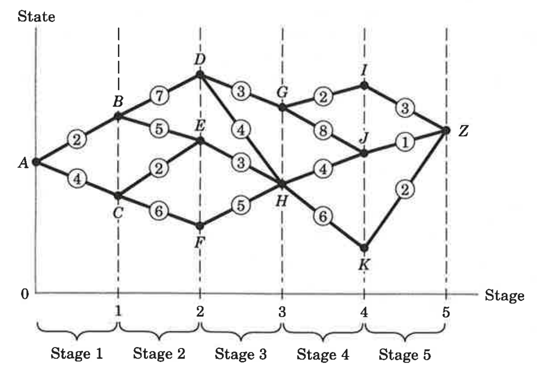
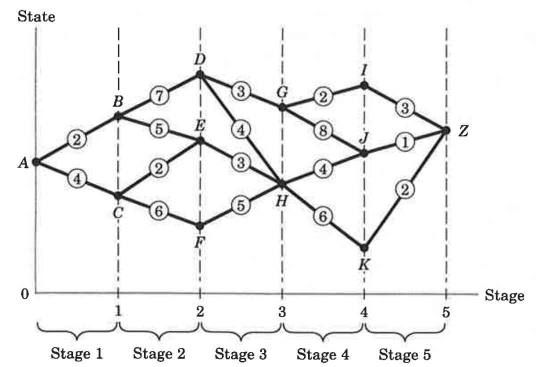


dynamic model: multistage decision making over time
available actions at A: $\{B, C\}$, etc.

cost (numbers in circles):
Which one is the cost-minimizing path?



Strategy 1: Starting at A, at each stage, choose the cost-min action at that stage: $\{B, E, H, J, Z\}$



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Problem: Strategy 1 dominated by Strategy 2: $\{C, E, H, J, Z\}$

cost of Strategy 1 = 15
cost of Strategy 2 = 14 $\Rightarrow$ Strategy 1 is not optimal

Lesson: A myopic, one-stage-at-a-time optimization procedure does not, in general, yield the optimal path!

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- Dynamic programming
- Calculus of variation
- Optimal control theory (Maximum principle)

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We will deal with optimal control theory

- continuous time
- infinite horizon problems
- autonomous problems
  (standard growth models)
- 1 choice variable and 1 state variable

Ingredients of Static Optimization

- Classical programming: ingredients

  choice variables $x = (x_1, x_2, x_3, ...x_N)$
  objective function $f(x)$
  equality constraint $0 = g(x)$

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  - solution technique: Lagrange function
    $\mathcal{L}(x, \mu) = f(x) + \mu\, g(x)$
  - interior solution satisfies necessary first-order conditions (FOC)

$$\frac{\partial \mathcal{L}(x, \mu)}{\partial x_i} = 0\,, \quad \frac{\partial \mathcal{L}(x, \mu)}{\partial \mu} = 0\,, \quad i = 1, ..., N$$

Query. Identify these ingredients in the standard utility maximization problem.

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Optimal control problem has same ingredients + a few others:

- variables are dated
- flow variables and stock variables
- constraint becomes dynamic (links variables across time)
- inital- and terminal conditions for stock variables

Digression: Flow variables versus stock variables

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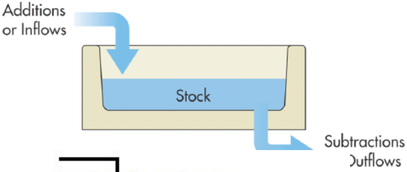
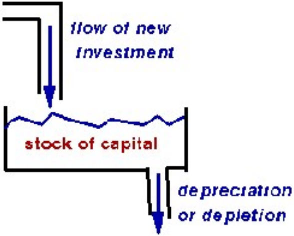
- stock: measured as a quantity at a given point in time (e.g., K)
- flow: measured as a quantity per unit of time (e.g., income, savings, consumption, investment)

- one cannot simply add stock and flow variables, as their units differ!

Two examples





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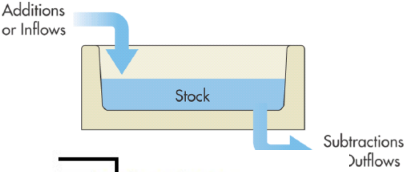

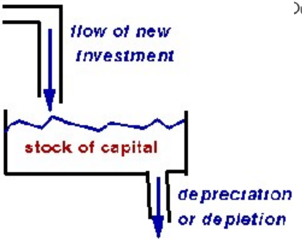

Query. Which of the following variables are stocks/flows?

savings, consumption, income, wealth, labor input, your assets, production

- Stocks and flows in optimal control theory

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    state variable changes in choice for control variable
    (think of bathtub example)

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  State variables make the optimization problem a dynamic problem!

  Without state variables, a myopic, one-stage-at-a-time optimization procedure is optimal.

Ingredients in Optimal Control Problems

- Ingredients (specialized version)
  - two types of (dated) variables:
    - choice variables x_t "control variables"

      quantity per period of time;
      time period needed to give variable a meaning
    - stock variables y_t

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  - terminal condition "transversality condition"
    What happens to y_t (or its value) as $t \to \infty$?
    $\to$ required to pick best path of $(x_t)_{t=0}^{\infty}$

Ingredients cont.'ed

Economic growth models: let $y_t = k_t^\alpha$ (*)

- *Query.* Suppose $Y_t = K_t^\alpha L_t^{1-\alpha}$, and $y_t \equiv Y_t/L_t$, $k_t \equiv K_t/L_t$. Derive (*)!

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  $\dot{k}_t = g(c_t, k_t) = y_t - c_t - \delta\, k_t = k_t^\alpha - c_t - \delta\, k_t$
- terminal = transversality condition

  whatever stock of capital is left over at the end of time, $t \to \infty$,
  must have no value according to the intertemporal objective function

Maximum Principle

- Problem: choose $(x_t)_{t=0}^{\infty}$ so as to max objective function

  s.t.

  $x_t \geq 0$

  $\dot{y}_t = g(x_t, y_t, t)$, y_0 given

  transversality condition

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- Solution technique: *current value* Hamiltonian function

  $\mathcal{H}(x_t, y_t, \mu_t) = f(x_t, y_t) + \mu_t \, g(x_t, y_t)$

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  $\mathcal{H}(x_t, y_t, \mu_t) = f(x_t, y_t) + \mu_t\, g(x_t, y_t)$

- Maximum Principle: At an interior solution, there exists a continuous μ_t so that $(x_t, y_t)_{t=0}^{\infty}$ satisfies first-order conditions

$$\frac{\partial \mathcal{H}(.)}{\partial x_t} = 0\,, \quad \frac{\partial \mathcal{H}(.)}{\partial \mu_t} = \dot{y}_t\,, \quad \frac{\partial \mathcal{H}(.)}{\partial y_t} = \rho\mu_t - \dot{\mu}_t$$

and the transversality condition $\lim_{t\to\infty} \mu_t\, y_t\, e^{-\rho\, t} = 0$.

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- Hamiltonian $\leftrightarrow$ Lagrangian
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  ensures dynamic constraint
- $\frac{\partial \mathcal{H}(.)}{\partial y_t} = \rho \mu_t - \dot{\mu}_t$
  marginal benefit of an additional unit of y_t "consumed" today on objective function

Transversality condition (TC)

- $\lim_{t\to\infty} \mu_t \, y_t \, e^{-\rho\, t} = 0$

  Query. Suppose y_t is some household's capital stock, and ρ is a discount rate. Give an intuitive interpretation of the transversality condition.

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- TC determines the initial value of the control variable

  (the initial value of the state variable is exogenously given)

Second Order Conditions

Maximum Principle provides necessary FOC

Sufficient conditions (Arrow)

- Consider $H^0(y,\mu) \equiv f(x^*, y) + \mu\, g(x^*, y)$

  If, for a given μ, $H^0(y,\mu)$ is concave in y for all t, the above conditions are also sufficient (not only necessary) for a global maximum.

Examples

A few hints for the examples

- notation: x_t means x as a function of time t in continuous time
- $\dot{x}_t = \frac{\partial x_t}{\partial t}$
- growth rate $\frac{\dot{x}_t}{x_t}$

Examples

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- notation: x_t means x as a function of time t in continuous time
- $\dot{x}_t = \frac{\partial x_t}{\partial t}$
- growth rate $\frac{\dot{x}_t}{x_t}$
- simple differential equation

$$\dot{y}_t + a y_t = b_t \,, \quad y_0 \text{ given} \,, \quad t \geq 0$$

$$y_t = y_0 e^{-a\,t} + \int_0^t b_\tau \, e^{-a(t-\tau)} \, d\tau$$

Example 1

Choose $(x_t)_{t=0}^{\infty}$ so as to

$$\max \int_0^\infty (1 - x_t^2) e^{-\rho t}\, dt$$

$$\text{s.t. } x_t \geq 0 \quad \text{(control region)}$$

$$\dot{y}_t = x_t\,, \quad y_0 > 0 \text{ given}$$

Set up the optimal control problem and solve for $(x_t, y_t, \mu_t)_{t=0}^{\infty}$.

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Remark. This is a degenerate problem. x impacts on y but y neither restricts (future) choices of x, nor does y show up in the objective function. So, y has no value, and we expect its shadow price $\mu_t = 0$.

Example 2 (Optimal oil extraction)

A firm extracts oil from an oil field.

y_t stock of oil in the oil field (number of barrels)
x_t rate of extraction per unit of time (barrels of oil)
q_t market price per barrel of oil
$c(x_t, y_t)$ cost of extraction per unit of time

The firm wants to extract oil so as to max present value of profits, $\pi(x_t, y_t)$, given the constant discount rate ρ.

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(a) Set up the optimal control problem.

(b) Suppose (i) $\dot{q}_t = 0$, (ii) $c(x_t, y_t) = \alpha\, x_t$. Calculate the optimal growth rate of extraction. Is it growing or declining over time? Why?

Example 3

Consider $\rho = 0$. Choose $(x_t)_{t=0}^{1}$ so as to

$$\max \int_0^1 -x_t^2 \, dt$$

$$\text{s.t. } \dot{y}_t = y_t + x_t$$

$$y_0 = 1 \,, \; y_1 = 0$$

Set up the optimal control problem and calculate the optimal $(x_t)_{t=0}^{1}$. Show that the sufficient SOC are satisfied.