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Application I. Ramsey Model

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The Ramsey – Cass – Koopmans Model

- Recap of optimal control theory
- Market conditions
- Agents (households, firms)
- Dynamic system
- Steady state (BGP), stability
- Saving rate
- Examples

Recap: Optimal control

- Main ingredients
  - control- and state variables (x_t, y_t)
  - instantaneous (date-t-) objective function $f(x_t, y_t)$
  - intertemporal objective function $\int_0^\infty f(x_t, y_t)\, e^{-\rho\, t}$
  - dynamic constraint $\dot{y}_t = g(x_t, y_t)$
  - initial- and end-of-time- (=transversality) conditions for the *state* variable

- Problem
  - $(x_t, y_t)_{t=0}^{\infty}$ is admissible if dynamic- and boundary constraints are satisfied
  - among all admissible paths for $(x_t, y_t)_{t=0}^{\infty}$ find the one(s) that maximize the intertemporal objective function $\int_0^{\infty} f(x_t, y_t)\, e^{-\rho\, t}$
  - such a pair $(x_t, y_t)_{t=0}^{\infty}$ is called *optimal* or a solution to the optimal control problem
  - for finding the solution, we employ the Hamiltonian function

- Procedure

1. Set up the (current value) Hamiltonian
   $H(x_t, y_t, \mu_t) = f(x_t, y_t) + \mu_t\, g(x_t, y_t)$
2. At every point in time, maximize $H(.)$ with respect to the control variable: $\frac{\partial H(.)}{\partial x_t} = 0$
3. Differentiate $H(.)$ w.r.t. the state variable and set the result equal to $(\rho\mu_t - \dot{\mu}_t)$. That is, $\frac{\partial H(.)}{\partial y_t} = (\rho\mu_t - \dot{\mu}_t)$ for all t.
4. Now, use the *Maximum Principle*. It says that for an interior solution, $(x_t, y_t)_{t=0}^{\infty}$, there exists a continuous function μ_t, such that for all t (2.) and (3.) hold, *and* the transversality condition $\lim_{t\to\infty} \mu_t\, y_t\, e^{-\rho\, t} = 0$ is satisfied.

Market Conditions

- Markets

  output ("all-purpose" output good)
  labor
  capital services (rental market for capital goods)

- Agents

  1 representative household (family) with $L_t = L_0 e^{nt} = e^{nt}$ members ("consumers")
  firms (technology) as in Solow model: CD technology

- 1 Asset: ownership claims on capital (K)

- No uncertainty

- Perfect competition

  prices exogenous to agents
  real prices are measured in terms of the output price
  CRS technology: no profits

Notation alert

- uppercase letters: aggregate variables (K_t, Y_t, C_t)
- lowercase letters: per capita variables ($k_t = K_t/L_t, y_t = Y_t/L_t, c_t = C_t/L_t$)
- level of technology A_t (index, number of patents)
  technical progress = growth rate of A_t, that is: $\dot{A}_t/A_t$
- effective labor input: $A_t L_t$
- lowercase-tilde-letters: per effective labor variables

$$\tilde{k}_t = K_t/(L_t A_t)$$
$$\tilde{y}_t = Y_t/(L_t A_t)$$
$$\tilde{c}_t = C_t/(L_t A_t)$$

Households

- Asset: capital $K_t = k_t L_t = \tilde{k}_t (A_t L_t)$
- Representative household belongs to family with L_t members

$$U_0 = \int_0^\infty (\ln c_t)\, L_t\, e^{-\rho t} dt = \int_0^\infty (\ln c_t)\, e^{-(\rho - n)t} dt$$

ρ	*pure* rate of time preference
$(\rho - n)$	*effective* rate of time preference

- Flow budget constraint

$$\dot{K}_t = r_t K_t + w_t L_t - c_t L_t\,, \quad K_0 \text{ given}$$

$$\dot{k}_t = (r_t - n) k_t + w_t - c_t\,, \quad k_0 \text{ given}$$

- Solvency (No-Ponzi-Game) condition (NPG)

$$\lim_{t\to\infty} K_t \, e^{-\int_0^t r_s \, ds} \geq 0$$

use $K_t = k_t L_t = k_t e^{n\,t}$, then:

$$\lim_{t\to\infty} k_t \, e^{-\int_0^t (r_s - n) \, ds} \geq 0$$

$r_t - n =$ growth-corrected rate of interest

- Representative consumer's problem:

choose $(c_t)_{t=0}^{\infty}$ so as to maximize U_0

subject to:

(i) $c_t \geq 0$
(ii) flow budget constraint
(iii) solvency condition

Optimization part

- Optimal control theory in the Ramsey model
  - two types of variables:
    - control variable c_t
    - stock variables k_t
  - objective function $\int_0^\infty (\ln c_t) e^{-(\rho-n)\,t}\,dt$
  - dynamic constraint $\dot{k}_t = (r_t - n)k_t + w_t - c_t$
  - transversality condition (TVC)
    $\lim_{t\to\infty} \mu_t\, k_t\, e^{-(\rho-n)\,t} = 0$

Solution procedure

1 set up *current-value* Hamiltonian

$$H(c_t, k_t, \mu_t) = (\ln c_t) + \mu_t[(r_t - n)k_t + w_t - c_t]$$

2 choose c_t so as to max Hamiltonian

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0 \qquad (1)$$

3 differentiate Hamiltonian w.r.t state variable

$$\frac{\partial H}{\partial k_t} = (r_t - n)\mu_t = (\rho - n)\,\mu_t - \dot{\mu}_t \Leftrightarrow -\frac{\dot{\mu}_t}{\mu_t} = r_t - \rho \qquad (2)$$

4 apply Maximum Principle
(for picking U_0-max consumption-/saving path)

$$\lim_{t\to\infty} \mu_t\, k_t\, e^{-(\rho-n)\,t} = 0 \qquad (3)$$

(1) $\frac{1}{c_t} = \mu_t$ MB = MC

(2) $r_t = \rho - \frac{\dot{\mu}_t}{\mu_t}$ return to savings = required return

(3) $\lim_{t\to\infty} \mu_t \, k_t \, e^{-(\rho - n)\, t} = 0$

$<$ violates NPG

$>$ k with positive contribution to utility "left over"

Remark: FOC (2) + TVC (3) imply NPG condition ▸ Proof

Ramsey

- Growth of per capita consumption

differentiate (1) w.r.t. time & divide both sides by μ_t

$$c_t^{-1} = \mu_t \Rightarrow -\frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho) \quad \text{by (2)}$$

$$\frac{\dot{c}_t}{c_t} = r_t - \rho$$

$\rightarrow$ Keynes-Ramsey rule

Firms

- Setup ($\rightarrow$ Solow model)

$$Y_t = K_t^{\alpha} \, (A_t \, L_t)^{1-\alpha}$$

$$\dot{A}_t / A_t = \gamma \geq 0$$

$$\dot{L}_t / L_t = n \geq 0$$

per unit of effective labor (AL)
$\tilde{y}_t = \tilde{k}_t^{\alpha}$

- Factor markets

$$\hat{r}_t = r_t + \delta = \partial \, Y_t / (\partial K_t) = \alpha \, \tilde{y}_t / \tilde{k}_t \Rightarrow r_t = \alpha \, \tilde{y}_t / \tilde{k}_t - \delta$$

$$w_t / A_t = \partial \, Y_t / (\partial L_t) = (1 - \alpha) \, \tilde{y}_t$$

- Capital accumulation

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

Dynamic system

- *Two* "dynamic variables": capital $\tilde{k}_t$, consumption $\tilde{c}_t$

$$\dot{K}_t = Y_t - C_t - \delta\, K_t$$

$$\Leftrightarrow \frac{\dot{K}_t}{K_t}\frac{K_t}{A_t L_t} = \frac{Y_t}{A_t L_t} - \frac{C_t}{A_t L_t} - \delta\,\frac{K_t}{A_t L_t}$$

$$\Leftrightarrow \frac{\dot{K}_t}{K_t}\tilde{k}_t = \tilde{y}_t - \tilde{c}_t - \delta\,\tilde{k}_t$$

as $$\frac{\dot{\tilde{k}}_t}{\tilde{k}_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} - (\gamma + n)$$

$$\Rightarrow \dot{\tilde{k}}_t = \tilde{y}_t - \tilde{c}_t - (\gamma + \delta + n)\,\tilde{k}_t$$

$$\Leftrightarrow \dot{\tilde{k}}_t = \tilde{k}_t^{\alpha} - \tilde{c}_t - (\gamma + \delta + n)\,\tilde{k}_t$$

Dynamic System

- Consumption

$$\frac{\dot{c}_t}{c_t} = (r_t - \rho) = [(\alpha\, \tilde{k}_t^{\alpha-1} - \delta) - \rho]$$

$$\text{as} \quad \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{\dot{c}_t}{c_t} - \gamma$$

$$\Rightarrow \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = [(\alpha\, \tilde{k}_t^{\alpha-1} - \delta) - \rho - \gamma]$$

- Dynamic system in $(\tilde{k}, \tilde{c})$

$$\dot{\tilde{k}}_t = \tilde{k}_t^{\alpha} - \tilde{c}_t - (\gamma + \delta + n)\, \tilde{k}_t \tag{4}$$

$$\dot{\tilde{c}}_t = [(\alpha\, \tilde{k}_t^{\alpha-1} - \delta) - \rho - \gamma]\, \tilde{c}_t \tag{5}$$

$$\tilde{k}_0 \text{ given} \quad ; \quad \text{TVC} \tag{6}$$

Grafic Analysis of Transitional Dynamics

- Idea: investigate the $\dot{\tilde{k}} = 0, \dot{\tilde{c}} = 0$-loci in $(\tilde{k}, \tilde{c})$-plane

$\dot{\tilde{k}} = 0$-locus: $\tilde{c} = \tilde{k}^{\alpha} - (\gamma + \delta + n)\,\tilde{k} \equiv \tilde{c}(\tilde{k})$

inversely U-shaped with maximum at $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = \gamma + n \quad \Rightarrow \quad \tilde{k}_{GR} = \left[\frac{\alpha}{\gamma + \delta + n}\right]^{1/(1-\alpha)}$$

points above $\tilde{c}(\tilde{k})$: $\dot{\tilde{k}} < 0$ (too much $\tilde{c}$); points below $\tilde{c}(\tilde{k})$: $\dot{\tilde{k}} > 0$

$\dot{\tilde{c}} = 0$-locus satisfies for all $\tilde{c}$:

vertical line at $\tilde{k}_{KR}$ (modified GR):

$$f'(\tilde{k}_{KR}) - \delta = \rho + \gamma \quad \Rightarrow \quad \tilde{k}_{KR} = \left[\frac{\alpha}{\rho + \gamma + \delta}\right]^{1/(1-\alpha)}$$

points to the right: $\dot{\tilde{c}} < 0$ (too low r); points to the left: $\dot{\tilde{c}} > 0$

Grafic Analysis of Transitional Dynamics

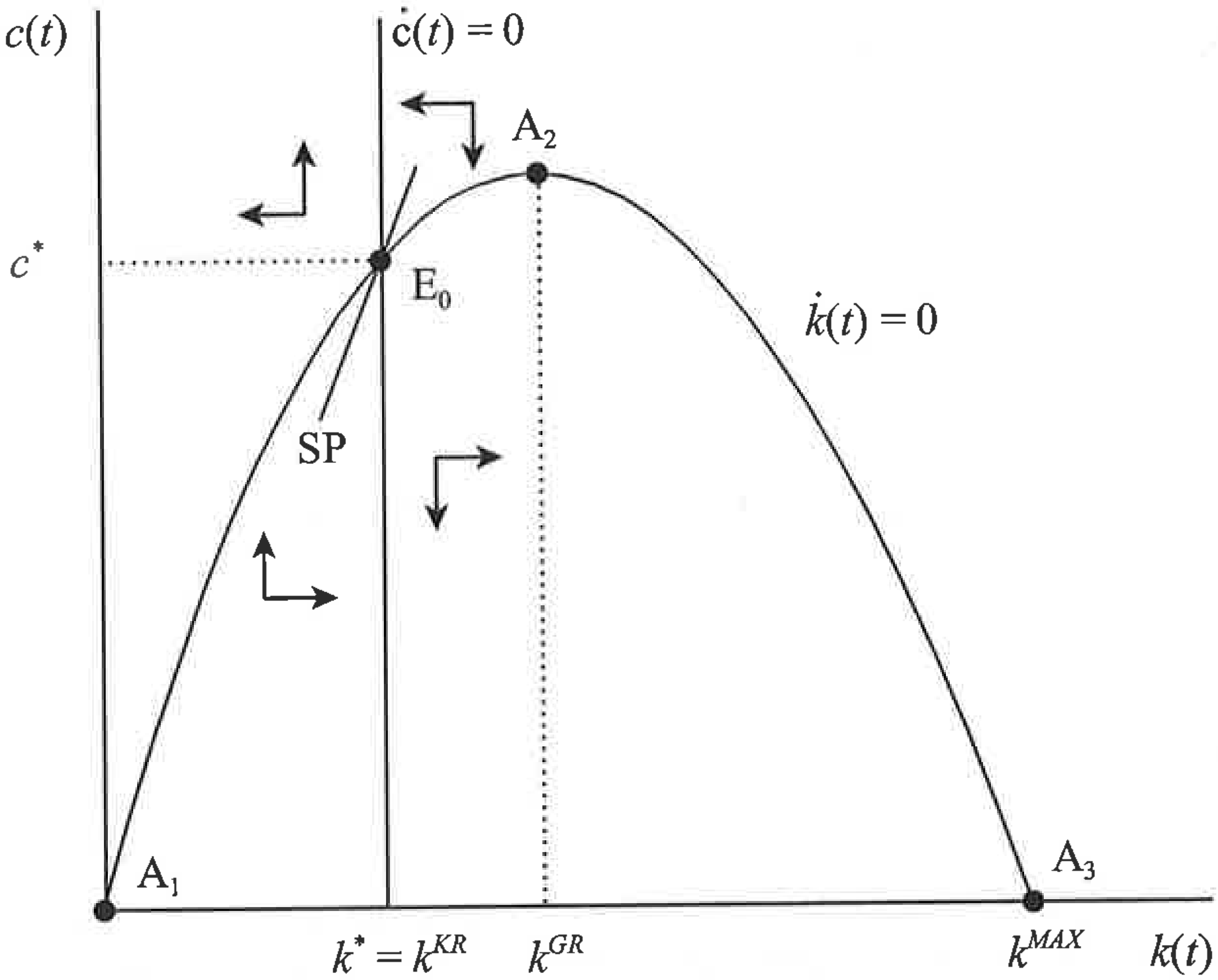

Figure 13.9: Phase diagram of the Ramsey model

BGP Rich/Poor?

Balanced Growth Path (steady state)

- BGP (steady state)

  (k, y, c) grow at constant rates (stylized facts)

  - from capital accumulation $\dot{k}/k = y/k - c/k - (\delta + n)$

    $\Rightarrow g_k = g_y = g_c$

  - from production $g_k = g_y \Rightarrow g_k = g_y = \gamma$

  - $(\tilde{k}, \tilde{y}, \tilde{c})$ constant

- BGP characterized by intersection of $\dot{\tilde{k}} = 0$, $\dot{\tilde{c}} = 0$-loci in $(\tilde{k}, \tilde{c})$-plane E_0 in the figure

Balanced Growth Path

- Existence

  TVC must be satisfied $\Leftrightarrow \rho + \gamma > \gamma + n$

  BGP $\tilde{k}_{KR} < \tilde{k}_{GR}$

- BGP

  $f'(\tilde{k}^*) - \delta = \rho + \gamma \Rightarrow$

$$\tilde{k}^* = \left[\frac{\alpha}{\rho + \gamma + \delta}\right]^{1/(1-\alpha)}, \quad \tilde{y}^* = (\tilde{k}^*)^\alpha$$

$$\tilde{c}^* = (\tilde{k}^*)^\alpha - (\gamma + \delta + n)\tilde{k}^*$$

Ramsey

- per capita growth γ (exogenous)
- per capita income

  saving rate

$$s^* = \frac{\tilde{y}^* - \tilde{c}^*}{\tilde{y}^*} = \frac{\alpha(\gamma + \delta + n)}{\delta + \rho + \gamma}$$

  per capita income in terms of saving rate

$$y_t^* = \left[\frac{s^*}{n + \gamma + \delta}\right]^{\alpha/(1-\alpha)} A_t$$

$\rightarrow$ "rich" if: A_t, s^* high; n low ($\rightarrow$ Solow model)

$\rightarrow$ theory of evolution and long-run level of s_t

s^* determined jointly by preference- and technology parameters

e.g., impatience (ρ)

Ramsey

Value added (w.r.t. Solow)

- endogenous saving rate
- household behavior responds to shocks (e.g., tax shocks)
- model opens up for welfare analysis
- today one workhorse model for analyzing economic policy

Examples

Example 1

In the basic setup of the Ramsey model, we used a logarithmic instantaneous utility function. If instead we employ a CIES-(constant intertemporal elasticity of substitution) utility function (see below), calculate the economy's growth rate, saving rate, and per capita income along a BGP. How does the intertemporal elasticity of substitution affect (g_c, s, y)?

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}, \quad \theta > 0$$

Example 2.

Let us return to the logarithmic utility function. Suppose, the following tax program is introduced. The government taxes capital income at some constant rate, τ_k and rebates the tax revenue in a lump sum fashion back to all households. How does this "tax reform" affect the saving rate and per capita income along a BGP?

Ramsey

- Remark: FOC (2) + TVC (3) imply NPG condition

  from (2), $\mu_t = \mu_0 \, e^{-\int_0^t r_s - \rho \, ds}, \quad \mu_0 \neq 0$

  into (3) + dividing by μ_0 yields

$$\lim_{t \to \infty} k_t \, e^{-\int_0^t (r_s - n) \, ds} = 0 \quad \Rightarrow \quad \lim_{t \to \infty} k_t \, e^{-\int_0^t (r_s - n) \, ds} \geq 0 \quad \text{(NPG)}$$

back